Lecture Notes on Partial Differential Equations

Boundary value problem in two dimensions

Circular boundary

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Summary: In this lecture we discuss a boundary value problem involving a circular boundary in a plane. We show how solutions of the Laplace equation can be obtained for regions inside (or outside) the circle.

Example: Laplace Equation In Plane Polar Coordinates

We will now solve the Laplace equation

$$\nabla^2 u(r,\phi) = 0 \tag{1}$$

in two dimensions in plane polar coordinates. Obtain the most general form of solution. We shall then show how to fix the unknown constants appearing in the solution obtained if we know value of $u(r,\phi)$ on a circle and determine the solution every where inside (or outside) the circle of radius R_0. For this purpose we assume that on the circle the boundary condition is given by

$$u(r = R_0, \phi) = f(\phi) \tag{2}$$

where $f(\phi)$ is a given function of ϕ and that $u(r,\phi)$ is finite everywhere inside the circle.

The Laplace equation written in plane polar coordinates (r,ϕ) takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} = 0 \tag{3}$$

Substitute $u(r,\phi) = R(r)P(\phi)$ in the Laplace equation Eq.(3) and divide by $R(r)P(\phi)$ and also multiply by r^2 to get

$$r^2\frac{1}{R}\frac{d^2R}{dr^2} + r\frac{1}{R}\frac{dR}{dr} + \frac{1}{P}\frac{d^2P}{d\phi^2} = 0 \tag{4}$$

This gives

$$\frac{d^2P}{d\phi^2} = \mu P \tag{5}$$

and

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \mu R = 0 \tag{6}$$

where μ is a constant. We shall take up the cases $\mu = 0$ and $\mu \neq 0$ separately.

Case $\mu = 0$ Eq.(5) has solution

$$P(\phi) = A_0 + B_0\phi \tag{7}$$

and Eq.(6) assumes the form

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} = 0 \tag{8}$$

or

$$r\frac{d}{dr}(r\frac{dR(r)}{dr}) = 0 \tag{9}$$

giving

$$r\frac{dR}{dr} = \text{constant}, \quad say, C_0 \tag{10}$$

This has the obvious solution

$$R(r) = C_0 \log r + D_0 \tag{11}$$

Case $\mu \neq 0$ In this case the Eq.(5) has solutions

$$P(\phi) = A\ \exp(\sqrt{\mu}\phi) + B\ \exp(-\sqrt{\mu}\phi) \tag{12}$$

The equation Eq.(6) for R(r) is easily solved to give

$$R(r) = Cr^{\sqrt{-\mu}} + Dr^{-\sqrt{-\mu}} \tag{13}$$

Requirement of single valuedness :

This completes the solution of the Laplace equation in polar coordinates. Usually one imposes the requirement that$u(r, \pi)$ be single valued. To understand this requirement note that, for arbitrary r, ϕ and $\phi + 2\pi$ correspond to the same point on the plane. Hence we must have

$$u(r, \phi) = u(r, \phi + 2\pi) \tag{14}$$

This implies that $P(\phi)$ must be a periodic function of ϕ with period 2π. Thus circular boundary

$$P(\phi) = P(\phi + 2\pi) \tag{15}$$

when expanded Eq.(15) becomes Eq.(16)

$$A\ \exp(\sqrt{\mu}\phi) + B\ \exp(-\sqrt{\mu}\phi) = A\ \exp(\sqrt{\mu}(\phi + 2\pi)) + B\ \exp(-\sqrt{\mu}(\phi + 2\pi)) \tag{16}$$

$$A\ \exp(\sqrt{\mu}\phi)[1-\exp(\sqrt{\mu}2\pi)] + B\ \exp(-\sqrt{\mu}\phi)[1-\exp(\sqrt{\mu}2\pi)] = 0 \tag{17}$$

The above relation must be satisfied for all values of ϕ. Due to the linear independence of $\exp(\sqrt{\mu}\phi)$ and $\exp(-\sqrt{\mu}\phi)$ for $\mu \neq 0$, Eq.(17) implies that the the expressions multiplying the two exponentials must be separately be zero. This is satisfied if

$$\exp(\sqrt{\mu}2\pi) = 1 \tag{18}$$

This restricts the allowed values of μ to

$$\sqrt{\mu} = in \tag{19}$$

where n is a non- zero integer. Noting that $P(\phi)$ becomes

$$P(\phi) = A\ \exp(in\phi) + B\ \exp(-in\phi) \tag{20}$$

or equivalently $P(\phi)$ can be taken to be

$$P(\phi) = acos(n\phi) + bsin(n\phi) \tag{21}$$

Using $-\mu = n^2$ in Eq.(13), and forming the superposition the most general solution can be cast in the form

$$u(r,\phi) = a_0 + b_0 \log r + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})(a_n \cos(n\phi) + b_n \sin(n\phi)) \tag{22}$$

Solution inside the circle

In this case we are looking for a solution finite near $r = 0$ and we must set $D_n = 0$ and $b_0 = 0$. This is the case when solution for of $u(r,\phi)$ inside the circle is needed. For the interior the solution can now be further simplified to the form

$$u(r,\phi) = a_0 + \sum_{n=1}^{\infty} r^n (a_n\ \cos(n\phi) + b_n\ \sin(n\phi)) \tag{23}$$

Here C_n has been set equal to 1. This does not mean any loss of generality because the constants a_n and b_n are as yet unknown.

The unknown coefficients a_n and b_n in Eq.(23) and can be determined by imposing the requirement that $u(r,\phi)$ reduce to the given function $f(\phi)$ on the circle

$$u(r = R_0, \phi) = f(\phi). \tag{24}$$

As usual this implies that,

$$a_n = \frac{1}{2\pi}\frac{1}{R_0^n}\int_0^{\pi} f(\phi)\cos(n\phi)d\phi. \tag{25}$$

$$b_n = \frac{1}{2\pi}\frac{1}{R_0^n}\int_0^{\pi} f(\phi)\sin(n\phi)d\phi. \tag{26}$$

Solution outside the circle:

The solution finite everywhere outside the circle is similarly obtained by setting the constants C_n and b_0 of Eq.(22) are to set equal to zero. The corresponding solution assumes the form

$$u(r,\phi) = a_0 + \sum_{n=1}^{\infty} r^{-n}(a_n \cos(n\phi) + b_n \sin(n\phi)) \tag{27}$$

Again, the unknown coefficients a_n and b_n in Eq.(27) can be determined by imposing the requirement that $u(r,\phi)$ reduce to the given function $f(\phi)$ on the circle

$$u(r = R_0, \phi) = f(\phi) \tag{28}$$

This implies that the coefficients a_n and b_n are given by

$$a_n = R_0^n \frac{1}{2\pi} \int_0^{\pi} f(\phi) \cos(n\phi) d\phi \tag{29}$$

and

$$b_n = R_0^n \frac{1}{2\pi} \int_0^{\pi} f(\phi) \sin(n\phi) d\phi \tag{30}$$

So far we have discussed only those examples which, by means of separation of variables, lead to solution in terms of Fourier series expansion. In the next lecture we shall now take up the important case of problems in three dimensions with spherical symmetry which leads to series expansion in terms of special functions.