Solved Example

cv-ymp-05001 **Taylor Expansion**

Setting up the solution

The branch cut for $z^p, (z-1)^q$ is given to be along the positive real axis so we take the definition of the two functions as

$$\begin{aligned} z^p &= r^p e^{ip\theta}, \qquad 0 < \theta < 2\pi, &(1)\\ (z-1)^q &= \rho^q e^{iq\phi}, \qquad 0 < \phi < 2\pi. &(2)\end{aligned}$$

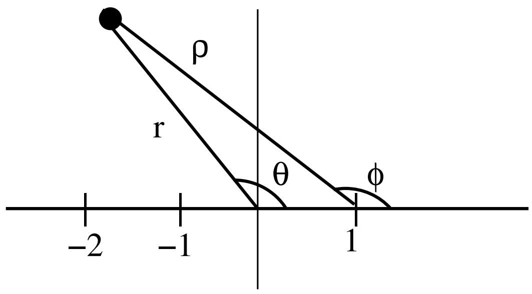

Fig. **1**

It is important that while computing the values of the functions and derivatives the values of θ, ϕ must be taken in the ranges as specified in (1) and (2). We shall find the Taylor series for the two functions z^p and $(z-1)^q$ separately and multiply the two series to find the series for the function $f(z)$.

Prepare for computing the values at the point -1:
From the definitions of r, θ, ρ and ϕ we see, from the figure that for the point $z_0 = -1$, we have

$$r = 1, \theta = \pi; \qquad \rho = 2, \phi = \pi.$$

Therefore

$$z_0 = |z_0| e^{i\pi} = e^{i\pi}; \qquad (z_0 - 1)^q = 2^q e^{iq\phi}. \tag{3}$$

Taylor series of z^p: We compute the successive derivatives at $z = z_0 = -1$

$$\begin{aligned} a_0 &= z^p\Big|_{z=-1} = e^{ip\pi} &(4)\\ a_1 &= \frac{dz^p}{dz}\Big|_{z=-1} = pz^{p-1}\Big|_{z=-1} = pe^{i(p-1)\pi} = -pe^{ip\pi} &(5)\\ a_2 &= \frac{1}{2}\frac{d^2z^p}{dz^2}\Big|_{z=-1} &(6)\\ &= \frac{1}{2}p(p-1)z^{p-2}\Big|_{z=-1} = \frac{1}{2}p(p-1)e^{ip\pi} &(7)\end{aligned}$$

Hence the Taylor series is given by

$$z^p = e^{ip\pi}\left[1 - pw - \frac{p(p-1)}{2}w^2 + ...\right] \tag{8}$$

where $w = z - z_0 = z + 1$.

Taylor series of $(z-1)^q$: The successive derivatives at $z=z_0$ are given by

$$a_0 = (z-1)^q\Big|_{z=-1} = 2^q e^{iq\pi} \tag{9}$$

$$a_1 = \frac{d(z-1)^q}{dz}\Big|_{z=-1} = qz^{q-1} = -q2^{q-1}e^{iq\pi} \tag{10}$$

$$a_2 = \frac{1}{2}\frac{d^2(z-1)^q}{dz^2}\Big|_{z=-1} \tag{11}$$

$$= \frac{1}{2}q(q-1)z^{q-1}\Big|_{z=-1} = 2^{q-2}\frac{q(q-1)}{2}e^{iq\pi} \tag{12}$$

Hence the Taylor series is given by

$$(z-1)^q = e^{iq\pi}2^q\Big[1-\frac{qw}{2}+\frac{q(q-1)}{8}w^2+...\Big] \tag{13}$$

(Note that we have taken 2^q out as a common factor.)

Taylor series of the product $z^p(z-1)^q$: We multiply the two Taylor series in (8) and (13) to get the final answer as

$$\begin{aligned} &z^p(z-1)^q \\ &= e^{ip\pi}\Big[1-pw-\frac{p(p-1)}{2}w^2+...\Big]\times e^{iq\pi}2^q\Big[1-\frac{qw}{2}+\frac{q(q-1)}{8}w^2+...\Big] \\ &= 2^q e^{i(p+q)\pi}\Big[1-(p+q/2)w+\Big(\frac{pq}{2}-\frac{p(p-1)}{2}+\frac{q(q-1)}{8}\Big)w^2+\dots\Big] \end{aligned} \tag{14}$$

For $p=1-q$ the series reduces to

$$\begin{aligned} &z^(1-q)(z-1)^q \\ &= 2^q e^{i(p+q)}\Big[1-(p+q/2)w-\Big(\frac{p(1-p)}{2}+\frac{p(p-1)}{2}-\frac{q(q-1)}{8}\Big)w^2+\dots\Big] \\ &= -2^q\Big(1+\frac{2-q}{2}w+\frac{q(q-1)}{8}w^2+\dots\Big) \end{aligned} \tag{15}$$