VS-01 Lessons in Vectors Spaces

Vector Spaces and Subspaces

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§1 Lesson Overview

Syllabus Vector Spaces; Subspace of a vector space

Prerequisites Basic set theory; Groups and fields

Lesson Objectives To define vector space and subspace; to illustrate the definitions with examples and counter examples.

§2 Lessons

§2.1 Vector Spaces

Definition 1 *Let* $\mathcal{F}$ *be a field and* $+$ *be a binary operation defined on a set* $\mathcal{V}$*. The triple* $\langle \mathcal{V}, +, \mathbb{F} \rangle$ *is a* **vector space** *on a field* $\mathcal{F}$ *if the following properties are satisfied.*

(V-1) To every pair of vectors $f, g \in \mathcal{V}$, there corresponds a vector $f + g \in \mathcal{V}$ called the sum of f and g such that

(i) $f + g = g + f \qquad \forall\ f, g \in \mathcal{V}$

(ii) $f + (g + h) = (f + g) + h \qquad \forall\ f, g, h \in \mathcal{V}$

(iii) $\exists$ a unique vector $0 \in \mathcal{V}$ such that

$$f + 0 = f \qquad \forall\ f \in \mathcal{V}$$

(iv) To every vector $f \in \mathcal{V}$, there corresponds a vector $-f \in \mathcal{V}$ such that

$$f + (-f) = 0$$

(V-2) $\forall\ \alpha \in \mathcal{F}$ and $f \in \mathcal{V}$ there corresponds a unique vector $\alpha f \in \mathcal{V}$ such that

$$\alpha(\beta f) = (\alpha\beta) f \qquad \forall\ \alpha, \beta \in \mathcal{F}$$

and

$$1.f = f \qquad \forall f \in \mathcal{V}$$

(V-3) $\forall \alpha, \beta \in \mathcal{F}$ and $\forall f, g \in \mathcal{V}$ we have

$$(\alpha + \beta) f = \alpha f + \beta f$$

and

$$\alpha(f + g) = \alpha f + \alpha g$$

Examples Of Vector Spaces

(I)
1. Every field $\mathcal{F}$ is also a vector space over $\mathcal{F}$ as field of scalars. Thus we have the following important special examples of vector spaces.
2. Set of all complex numbers $\mathbb{C}$ is a complex vector space with $\mathbb{C}$ as the field of scalars.
3. Set of all real numbers $\mathbb{R}$ is a real vector space with $\mathbb{R}$ as the field of scalars.
4. Set of all rational numbers $\mathbb{Q}$ is a rational vector space with $\mathbb{Q}$ as the field of scalars.

(II) Set of all n-tuples $(\alpha_1, \alpha_2, ..., \alpha_n)$ where $\alpha_k \in \mathcal{F}$ is denoted by $\mathcal{F}^n$. This set is vector space with $\mathcal{F}$ as field of scalars. Thus

1. $\mathbb{C}^n$ is a complex vector space over $\mathbb{C}$ as the field of scalars.
2. $\mathbb{R}^n$ is a real vector space over $\mathbb{R}$ as the field of scalars.
3. $\mathbb{Q}^n$ is a rational vector space over $\mathbb{Q}$ as the field of scalars.

(III)
1. All polynomials in a variable t, with coefficients in any field $\mathcal{F}$ is vector space $\mathscr{P}$.

   $$\mathscr{P} = \{p(t) | p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t + ... + \alpha_n t^n + ...\text{and } \alpha_j \in \mathcal{F}\}$$

   Here $\mathcal{F}$ can be any of the fields such as $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \ldots$
2. Consider the set $\mathscr{P}$ of all polynomials in a variable t, with coefficients in any field $\mathscr{F}$ and consider the subset $\mathscr{P}_N$ consisting of all polynomials of degree $\leq N$. Then $\mathscr{P}_N$ is a vector space.

(IV)
1. Let $\mathcal{F}$ be set of all functions defined on an interval $[a, b]$ and having complex values. With any one of the fields $\mathbb{C}, \mathbb{R}$, or $\mathscr{Q}$, $\mathcal{F}$ is a vector space.
2. Let $\mathcal{F}$ be as in (IV-1) and $\mathscr{C}^{(0)}$ be the subset of all continuous functions. Then $C^{(0)}$ is a vector space.
3. Let $\mathcal{F}$ be as in (IV-1) and $C^{(r)}$ be the subset of all functions for which $r-$ derivatives exist and are continuous on $[a, b]$. The $C^{(r)}$ is a vector space.
4. Let $\mathscr{C}^{(0)}$ be as in (IV-2). Let $\mathcal{S}$ be a subset of $\mathscr{C}^{(0)}$ consisting of those functions which vanish at a given point x_0.Then $\mathcal{S}$ is vector space. In general, if one can takes all functions which vanish at $x_1, x_2, \ldots, x_n$ then also we get a vector space.

(V) Let $\mathbb{M}_N$ be the set of all $N \times N$ matrices whose element are scalars from a field $\mathcal{F}$. With standard matrix addition as vector addition $\mathbb{M}_N$ is a vector space over the same field F

(VI) The set of all functions f on an interval $[a, b]$, for which $\int_a^b |f(x)|^p \, dx$ is finite, is a vector space denoted by $\mathscr{L}^p[a, b]$. That addition of two functions in $\mathscr{L}^p[a, b]$ gives back a function in the same space will not be proved here. The space $\mathscr{L}^p[a, b]$, for $p = 2$, is the set of all square integrable functions on the interval $[a, b]$.

(VII) The set of all infinite sequences $(\alpha_1, \alpha_2, \ldots, ..)$, such that the infinite series

$$\sum_{k=1}^{\infty} |\alpha_k|^p$$

converges, is a vector space denoted by ℓ^p. That the sum of two sequences, $\alpha, \beta \in \ell^p$ is also in ℓ^p, space requires a proof which will not be given here.

(VIII) A set $\{0\}$, consisting of only one element, the null vector, is a vector space over any field.

§2.2 Subspaces

Definition 2 *Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$. Let $\mathscr{S}$ be a subset of $\mathcal{V}$. Let the vector addition in $\mathscr{S}$ be defined in the same way as in $\mathcal{V}$. If $\mathscr{S}$ is also vector space over the same field $\mathcal{F}$, we say that $\mathscr{S}$ is* **subspace** *of $\mathcal{V}$.*

Examples Of Subspaces

1. Every vector space $\mathcal{V}$ is subspace of itself.
2. The subset having only the null vector, 0, is a subspace of every vector space.
3. Let $\mathcal{V}_1$ be the vector space of complex numbers over the field of real numbers. Let $\mathcal{V}_2$ be the vector space of all real numbers with $\mathbb{R}$ as the field of scalars. The $\mathcal{V}_2$ is a subspace of $\mathcal{V}_1$.

4. The set $C^{(1)}$ of functions with continuous first derivative is a subspace of the vector space of all continuous functions with the same field of scalars.

5. Let $C^{(0)}$[a,b] be the set of all continuous complex valued functions on the interval $[a, b]$. This set is a vector space and we have

   (a) the subset consisting of of all functions which vanish at a given point x_0 is a subspace.

   (b) the subset of $C^{(0)}$ consisting of all functions having value 1/2 at a point x_0 is not a subspace.

   (c) The set of all solutions of a linear differential equation

   $$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + y(x) = 0$$

   is a vector space.

6. Consider the set of all vectors in three dimensions, $\mathbb{R}^3$ which is real vector space. The subset S_1 of all vectors which are multiples of a fixed vector $\vec{A}$ and the subset S_2 of all vectors in a given fixed plane passing through the origin, and are two examples of subspaces of $\mathbb{R}^3$.

It is easy to see that intersection of two subspaces of a vector space is again a subspace.

§3 EndNotes

References

1. Halmos P. R. *Finite Dimensional Vector Spaces* Springer Verlag, East West Edition (1974).

2. Fraleigh J. B. *A First Course in Abstract Algebra*, Pearson Education Limited, Essex, (2014).