# VS-01 Lessons in Vectors Spaces

Groups and Fields

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## Contents

**§1 Lesson Overview** **1**

**§2 Lessons** **1**
§2.1 Groups . . . . . . . . . . 1
§2.2 Fields . . . . . . . . . . 2

**§3 EndNotes** **3**

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## §1 Lesson Overview

**Syllabus** Groups, Fields

**Lesson Objectives** You will learn definition of group and field with some examples and counter examples.

**Prerequisite** Basic set theory, binary operation

## §2 Lessons

### §2.1 Groups

**Definition 1** *To every ordered pair $\langle a, b\rangle$ of elements of a set $\mathcal{X}$ a* **binary operation** *assigns an element, denoted by $a * b$, of the set $\mathcal{X}$. For a binary operation to be a valid one it must be defined for all pairs and the $a * b$ must belong to the set and the result of binary operation must be unique.*

**Definition 2** *A* **group** *is a pair $\langle \mathcal{G}, *\rangle$ with a binary operation $*$ defined on a set $\mathcal{G}$ such that the following properties.*

(G-1) Associative property : $a * (b * c) = (a * b) * c \qquad \forall\ a, b, c \in \mathcal{G}$

(G-2) Existence of identity : $\exists$ an element $e \in \mathcal{G}$ such that

$$e * a = a * e = a \qquad \forall\ a \in \mathcal{G}.$$

(G-3) Existence of inverse : $\forall\ a \in \mathcal{G}$ there exists an element $a'$ such that

$$a * a' = a' * a = e$$

**Examples Of Groups**

1. The set of all real numbers $\mathbb{R}$ forms a group with addition as the binary operation.
2. The set of all complex numbers $\mathbb{C}$ is a group with addition as group operation.
3. The set of all positive, non-zero, real numbers $\mathbb{R}^+$ is a group with respect to multiplication as group operation.
4. The set of all $N \times N$ real ( or complex ) matrices form a group under matrix addition.
5. The group of all $N \times N$ real (or complex) matrices with determinant $\neq 0$ form a group under the matrix multiplication.

## §2.2 Fields

**Definition 3** *A* **field** $\mathcal{F}$ *is a triple* $\langle \mathcal{F}, +, \cdot \rangle$*, where,* $\cdot$ *and* $+$ *are two binary operations defined on a set* $\mathcal{F}$ *such that the axioms (F-I) to (F-III), given below, are satisfied. The elements of the field will be called* **scalars** *and will be denoted by greek letters* $\alpha, \beta, \gamma, \ldots$*.*

(F-1) To every pair $\alpha, \beta$ the scalar $\alpha + \beta$ is called the **sum** of $\alpha, \beta$ which satsifies the following axioms $\forall \alpha, \beta, \gamma \in \mathcal{F}$

(i) $\alpha + \beta = \beta + \alpha$
(ii) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
(iii) $\exists$ a unique scalar 0 such that $0 + \alpha = \alpha = \alpha + 0$
(iv) $\forall \alpha \in \mathcal{F}\ \exists$ a unique scalar $(-\alpha) \in \mathcal{F}$ we have $\alpha + (-\alpha) = 0$

These properties imply that $\mathcal{F}$ is a group with $+$ as binary operation.

(F-2) The scalar $\alpha \cdot \beta$ will be called the **product** of $\alpha, \beta$ and has the following properties.

(i) Commutative Property : $\alpha \cdot \beta = \beta \cdot \alpha$
(ii) Associative Property : $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
(iii) Existence of multiplicative identity : $\exists$ a unique scalar 1 such that

$$\alpha.1 = 1.\alpha = \alpha$$

(iv) $\forall \alpha \neq 0\ \exists$ a scalar denoted by $\alpha^{-1}$ such that $\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1$

(F-3) The sum and the product obey the distributive property :

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

**Examples Of Fields**

1. Set of all rational numbers $\mathbb{Q}$ is a field with usual addition and multiplication as the two binary operations.
2. Set of all real numbers $\mathbb{R}$ is a field with usual addition and multiplication as the two binary operations.
3. Set of all complex numbers $\mathbb{C}$ is a field with usual addition and multiplication as the two binary operations.
4. The set $\mathbb{Z}^+$ of all positive integers is not a field with the usual addition and multiplication as two binary operations ( give all possible reasons).
5. The set $\mathbb{Z}$ of all integers is not a field with the usual addition and multiplication. ( Give one reason ).

# §3 EndNotes

**References**


1. Halmos P. R. *Finite Dimensional Vector Spaces* Springer Verlag, East West Edition (1974).
2. Fraleigh J. B. *A First Course in Abstract Algebra*, Pearson Education Limited, Essex, (2014).