MP-07 Ordinary Differential Equations

Four Cases of Series Solution

A Bundle of Four Lectures

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Contents

§1 The Frobenius Method

In this method of series solution, for nth order linear ordinary differential equations, one starts with a trial solution of the form

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{1}$$

The trial solution is substituted in the differential equation $Ly = 0$and we demand that the coefficient of each power of x be zero. The resulting equations determine the index c and the coefficients a_n. At first we shall discuss the method by means of examples. Later we shall discuss a theorem which tell us the conditions under which this method will give rise to an n linearly independent solutions. The relevant theorem, known as Fuch's theorem also tell us the minimum radius of convergence of the solution obtained in the series form. We shall discuss only second order linear ordinary differential equations. To introduce the method we take up the Bessel's equation as an example. The Bessel's equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0 \tag{2}$$

Substituting Eq.(135)in Eq.(109) we get

$$x^2\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} + x\sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} + (x^2-\nu^2)\sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{3}$$

Rewriting Eq.(110) as

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c} + \sum_{n=0}^{\infty} a_n(n+c)x^{n+c} + \sum_{n=0}^{\infty} a_n x^{n+c+2} - \nu^2\sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{4}$$

we see that the lowest power of x in the above equation is x^c. Equating the coefficient of x^c in Eq.(111)to zero we get

$$a_0c(c-1) + a_0c - \nu^2 a_0 = 0 \tag{5}$$

$$a_0(c^2 - \nu^2) = 0 \tag{6}$$

Assuming $a_0 \neq 0$we get

$$c^2 - \nu^2 = 0 \tag{7}$$

This equation determine the index c and is called the indicial equation. For the present case the two possible values of are c_1 and c_2 where

$$c_1 = -\nu,\ c_2 = \nu \tag{8}$$

The details of the method of series solution depend on the roots of the indicial equation. For the second order differential equations under discussion the following cases arise

CASE-I : The roots of indicial equation are distinct and the difference of the roots is not an integer.

CASE-II : The roots of indicial equation are equal.

CASE-III : The difference of the roots of indicial equation is a non-zero integer and some coefficient a_n becomes infinite.

CASE-IV : The difference of the roots of indicial equation is a non-zero integer and some coefficient a_n becomes indeterminate.

We shall discuss the above four cases by means of separate set of examples. The series solution for the Bessel's equation is left as an exercise for the reader.

§2 Case-I :: Difference of Roots of Indicial Equation is not and Integer

EXAMPLE - I :

In this lecture we shall take up solution of an ordinary differential equation by the method of series solution. The example to be discussed is such that the indicial equation has two distinct roots and the difference of the roots is not an integer.
Consider the equation

$$4x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0 \tag{9}$$

Let us assume a solution in the form

$$y(x,c)=\sum_{n=0}^{\infty}a_nx^{n+c} \tag{10}$$

where c and a_n are to be fixed. Substituting Eq.(117) in the differential Eq.(116) we get

$$4x\sum_{n=0}^{\infty}a_n(n+c)(n+c-1)x^{n+c-2}+2\sum_{n=0}^{\infty}a_n(n+c)x^{n+c-1}+\sum_{n=0}^{\infty}a_nx^{n+c}=0 \tag{11}$$

or,

$$\sum_{n=0}^{\infty}4a_n(n+c)(n+c-1)x^{n+c-1}+\sum_{n=0}^{\infty}2a_n(n+c)x^{n+c-1}+\sum_{n=0}^{\infty}a_nx^{n+c}=0 \tag{12}$$

or,

$$\sum_{n=0}^{\infty}2(n+c)(2n+2c-1)a_nx^{n+c-1}+\sum_{n=0}^{\infty}a_nx^{n+c}=0 \tag{13}$$

We now equate coefficients of different powers of x to zero. The minimum power of x in Eq.(120) is x^{c-1}. So we get

$$\text{Coeff of } x^{c-1}: \quad a_0 2c(2c-1)=0 \tag{14}$$

$$\text{Coeff of } x^{c}: \quad a_1 2(c+1)(2c+1)+a_0=0 \tag{15}$$

$$\text{or,} \quad a_1=-\frac{a_0}{2(c+1)(2c+1)} \tag{16}$$

$$\text{Coeff of } x^{c+1}: \quad a_2 2(2+c)(4+2c-1)+a_1=0 \tag{17}$$

$$\text{or,} \quad a_2=-\frac{a_1}{(2c+3)(2c+4)} \tag{18}$$

$$\text{Coeff of } x^{c+m}: a_{m+1}2(m+c+1)(2m+2c+1)+a_m=0 \tag{19}$$

$$\text{or} \quad a_{m+1}=-a_m\ \frac{1}{2(m+c+1)(2m+2c+1)} \tag{20}$$

The Eq.(121) gives the indicial equation

$$2c(2c-1)=0 \tag{21}$$

$$\text{or,} \ \ c=0,\frac{1}{2} \tag{22}$$

Solution for c = 0

The recurrence relation Eq.(127) becomes

$$a_{m+1} = -\frac{1}{(2m+2)(2m+1)} a_m \tag{23}$$

$$\text{Therefore,} \qquad a_1 = -\frac{1}{2 \cdot 1} a_0 \tag{24}$$

$$a_2 = -\frac{1}{4 \cdot 3} a_1 \quad = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} a_0 \tag{25}$$

and

$$a_3 = -\frac{1}{6 \cdot 5} a_2 \quad = -\frac{1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} a_0 \tag{26}$$

$$\text{Thus } a_m = \frac{(-1)^m}{(2m)!} a_0 \tag{27}$$

and one solution for, $c = 0$, is

$$y_{\mathrm{I}}(x) = x^c \sum_{n=0}^{\infty} a_n x^n = a_0 \left\{ 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots + \frac{(-1)^m x^m}{(2m)!} + \ldots \right\} \tag{28}$$

Solution for $c = \frac{1}{2}$

In this case we have

$$a_{m+1} = -\frac{1}{(2m+3)(2m+2)} a_m \tag{29}$$

Therefore,

$$a_1 = -\frac{1}{3 \cdot 2} a_0 \tag{30}$$

$$a_2 = -\frac{1}{5 \cdot 4} a_1 = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} a_0 \tag{31}$$

$$a_3 = -\frac{1}{7 \cdot 6} a_2 = -\frac{1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} a_0 \tag{32}$$

In general,

$$a_m = \frac{(-1)^m}{(2m+1)!} a_0 \tag{33}$$

The second solution is, therefore, given by

$$y_{\mathrm{II}} = x^c \sum_{n=0}^{\infty} a_n x^n = a_0 \sqrt{x} \left\{ 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \ldots + \frac{(-1)^m x^m}{(2m+1)!} + \ldots \right\} \tag{34}$$

The most general solution of the differential equation (116) is given by

$$y(x) = \alpha y_{\mathrm{I}}(x) + \beta y_{\mathrm{II}}(x) \tag{35}$$

§3 Case-II:: Roots of Indicial Equation are equal

In this method we shall take up solution of an ordinary differential equation by the method of series solution. In this chapter we discuss two examples for which the indicial equation has two equal roots.

Case II : Example

The first example is the differential equation $Ly = 0$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \tag{36}$$

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{37}$$

$$\frac{d}{dx}y(x,c) = \sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} \tag{38}$$

$$\frac{d^2}{dx^2}y(x,c) = \sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} \tag{39}$$

Substituting Eq.(109) Eq.(110)Eq.(111) in the differential equation Eq.(135) gives

$$x\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} + \sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{40}$$

or,

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-1} + \sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{41}$$

or,

$$\sum_{n=0}^{\infty} a_n(n+c)^2 x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{42}$$

Before we start equating the coefficients of different powers of x to zero, we derive a result for later use (see Eq.(117)below).

We split off the $n = 0$ term from the remaining series in the first term in Eq.(114) and rewrite the l.h.s of Eq.(114) as

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = a_0c^2x^{c-1} + \sum_{n=1}^{\infty} a_n(n+c)^2x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} \tag{43}$$

In the first summation in the r.h.s. we replace n with $m+1$ and sum over m from 0 to ∞; while in the second summation we simply replace $n$with$m$. This gives

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = a_0c^2x^{c-1} + \sum_{m=0}^{\infty} a_{m+1}(m+c+1)^2x^{m+c} + \sum_{m=0}^{\infty} a_m x^{m+c} \tag{44}$$

or

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = a_0c^2x^{c-1} + \sum_{m=0}^{\infty} \left[a_{m+1}(m+c+1)^2 + a_m\right] x^{m+c} \tag{45}$$

Coming back to Eq.(114) we now equate coefficients of different powers of x to zero. The minimum power of x in Eq.(114)is x^{c-1}. So we get

$$\text{Coefficient of } x^{c-1}\text{: } a_0 c^2 = 0 \tag{46}$$

or, the roots of the indicial equation are coincident and

$$c = 0 \tag{47}$$

$$\text{Coeff of } x^c : a_1(c+1)^2 + a_0 = 0 \tag{48}$$

$$a_1 = -\frac{a_0}{(c+1)^2} \tag{49}$$

$$\text{Coeff of } x^{c+1} : a_2(2+c)^2 + a_1 = 0 \tag{50}$$

$$\text{or, } a_2 = -\frac{a_0}{(c+2)^2(c+1)^2} \tag{51}$$

$$\text{Coeff of } x^{c+m} : a_{m+1} = -\frac{a_m}{(m+c+1)^2} \tag{52}$$

This gives

$$a_m = (-1)^m \frac{a_0}{[(c+m)(c+m-1)\dots(c+1)]^2} \tag{53}$$

and hence from (109)

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c} = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+c}}{[(c+n)(c+n-1)\dots(c+1)]^2} \tag{54}$$

Notice that, if we use Eq.(126) in Eq.(117), one gets that for $c \neq 0$ $y(x,c)$ satisfies the relation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = a_0 c^2 x^{c-1} \tag{55}$$

This Eq.(127) shows that the two solutions of the given differential equation are

$$y_{\text{I}}(x,c) = y(x,c)|_{c=0} \text{ and } y_{\text{II}}(x,c) = \left.\frac{dy(x,c)}{dc}\right|_{c=0} \tag{56}$$

We shall now determine the series for the two solutions in Eq.(128). The coefficients a_m can be expressed in terms of gamma functions $\Gamma(x)$ making use of the property

$$\Gamma(z+1) = z\Gamma(z) \tag{57}$$

Using Eq.(129) repeatedly we get, for $r < n$,

$$\begin{aligned} \Gamma(z+n+1) &= (z+n)\Gamma(z+n) && (58)\\ &= (z+n)(z+n-1)\Gamma(z+n-1) && (59)\\ &= \dots\dots \\ &= (z+n)(z+n-1)(z+n-2)\dots(z+r)\Gamma(z+r) \end{aligned}$$

Or

$$(z+n)(z+n-1)\dots(z+r) = \frac{\Gamma(z+n+1)}{\Gamma(z+r)} \tag{60}$$

Using Eq.(132) with $z = c$, $r = 1$we get

$$(c+n)(c+n-1)\ldots\ldots(c+1) = \frac{\Gamma(c+n+1)}{\Gamma(c+1)} \tag{61}$$

Use Eq.(133) to rewrite Eq.(126) to get

$$y(x,c) = a_0[\Gamma(c+1)]^2 \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+c}}{[\Gamma(c+n+1)]^2} \tag{62}$$

Setting $a_0[\Gamma(c+1)]^2 = 1$, we get

$$y(x,c) = x^c \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{[\Gamma(c+n+1)]^2} \tag{63}$$

The two solutions of the given differential equation are

$$+\, y_1(x) = y(x,c)|_{c=0} = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{[\Gamma(n+1)]^2} = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{(n!)^2} \tag{64}$$

and

$$y_2(x,c) = \left.\frac{dy(x,c)}{dc}\right|_{c=0} \tag{65}$$

For the second solution the derivative w.r.t. c at $c = 0$ is needed and can be conveniently expressed in terms of the $\Gamma(x)$ and the function $\psi(x)$, where

$$\psi(x) = \frac{1}{\Gamma(x)}\frac{d\Gamma(x)}{dx} = \frac{d}{dx}\log\Gamma(x) \tag{66}$$

Therefore, computing the derivative of $\frac{1}{[\Gamma(x)]^2}$

$$\frac{d}{dx}\frac{1}{[\Gamma(x)]^2} = -2\frac{1}{[\Gamma(x)]^3}\frac{d\Gamma(x)}{dx} = -2\frac{\psi(x)}{[\Gamma(x)]^2} \tag{67}$$

Differentiating $y(x,c)$ given by Eq.(100) we get

$$\frac{dy(x,c)}{dc} = x^c \log x \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{[\Gamma(c+n+1)]^2} + x^c \sum_{n=0}^{\infty}(-1)^n \frac{d}{dc}\frac{1}{[\Gamma(c+n+1)]^2} \tag{68}$$

Hence

$$y_2(x) = \left.\frac{dy(x,c)}{dc}\right|_{c=0} = y_1(x)\log x - 2\sum_{n=0}^{\infty}(-1)^n\, x^n\, \frac{\psi(n+1)}{[\Gamma(n+1)]^2} \tag{69}$$

In the last step in Eq.(69) we have used Eq.(104) to get

$$\left.\frac{d}{dc}\frac{1}{[\Gamma(c+n+1)]^2}\right|_{c=0} = -2\,\frac{\psi(n+1)}{[\Gamma(n+1)]^2} = -2\frac{\psi(n+1)}{(n!)^2} \tag{70}$$

The most general solution is a linear combination of $y_1(x)$ and $y_2(x)$

$$y(x) = \alpha\, y_1(x) + \beta\, y_2(x) \tag{71}$$

Question: In going from Eq.(134)to Eq.(100) we have made a choice

$$a_0 = \frac{1}{[\Gamma(c+1)]^2}$$

How will the solution y_2 change if we had proceed without making this choice ? It can be verified that the most general form Eq.(71) of the solution is not affected by this choice

§4 Case-III:: Roots Differ by an Integer and Some Coefficient is Infinite

We shall now take up the series solution for differential equations when the roots of the indicial equation differ by an integer $\neq 0$. For such equations two different possibilities arise.The first possibility, discussed in this lecture is that roots of the indicial equation differ by an integer and this results in some coefficient becoming infinite. In the other possibility, to be taken up in the next lecture, is when some coefficient becomes indeterminate.

Case-III : Example

An example of the case-III is the ordinary differential equation $Ly = 0$ where

$$Ly = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (2x+1)y \tag{72}$$

Let

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{73}$$

Then we have

$$\frac{d}{dx}y(x,c) = \sum_{n=0}^{\infty} a_n (n+c) x^{n+c-1} \tag{74}$$

$$\frac{d^2}{dx^2}y(x,c) = \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} \tag{75}$$

Substituting in the given differential equation

$$\begin{aligned} x^2 \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} + \\ x\sum_{n=0}^{\infty} a_n (n+c) x^{n+c-1} - (2x+1)\sum_{n=0}^{\infty} a_n x^{n+c} = 0 \end{aligned} \tag{76}$$

Or we have,

$$\begin{aligned} \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c} \quad &+ \quad x\sum_{n=0}^{\infty} a_n (n+c) x^{n+c-1} \\ &- \quad (2x+1)\sum_{n=0}^{\infty} a_n x^{n+c} \;=\; 0 \end{aligned} \tag{77}$$

This can be rearranged as

$$\sum_{n=0}^{\infty} a_n \left\{(n+c)^2 - 1\right\} x^{n+c} - 2\sum_{n=0}^{\infty} a_n x^{n+c+1} = 0 \tag{78}$$

Before we start equating the coefficients of different powers of x to zero, we derive a result for (see Eq.(117) below) for later use. We split off the $n = 0$ term in the first sum and write it separately.

$$a_0(c^2-1)x^c + \sum_{n=0}^{\infty} a_n \left\{(n+c)^2 - 1\right\} x^{n+c} - 2\sum_{n=0}^{\infty} a_n x^{n+c+1} = 0 \tag{79}$$

The summation index in the second sum can be redefined from n to $r = n+1$, so that sum over r runs from 0 to ∞. Thus we get

$$a_0(c^2-1)x^c + \sum_{r=0}^{\infty} a_{r+1}\left\{(r+1+c)^2-1\right\}x^{r+1+c} - 2\sum_{r=0}^{\infty} a_r x^{r+c+1} = 0 \tag{80}$$

The left hand side of Eq.(114) is just $Ly(x,c)$. Eq.(116) enables us to rewrite $Ly(x,c)$ as

$$Ly(x,c) = a_0(c^2-1)x^c + \sum_{r=0}^{\infty} a_{r+1}\left\{[(r+1+c)^2-1] - 2a_r\right\}x^{r+1+c} = 0 \tag{81}$$

Eq.(117) will be needed below, for the moment we get back to Eq.(114). The minimum power of x in Eq.(116) is x^c, and its coefficient is equated to zero to get

$$a_0(c^2-1) = 0. \tag{82}$$

The indicial equation is, therefore, given by $c^2-1=0$ and the possible values of c are ± 1. Equating the coefficients of successive powers x^{c+1}, x^{c+2} etc. to zero gives

$$\text{Coefficient of } x^{c+1}: \quad a_1\left[(c+1)^2-1\right] - 2a_0 = 0 \tag{83}$$

$$\text{therefore} \quad a_1 = \frac{2a_0}{c(c+2)} \tag{84}$$

$$\text{Coefficient of } x^{c+2}: a_2[(c+2)^2-1] - 2a_1 = 0 \tag{85}$$

and hence

$$a_2 = \frac{2a_1}{(c+1)(c+3)} = \frac{2.2a_0}{(c+1)(c+3)(c+2)c} \tag{86}$$

Note that the coefficient a_2 becomes infinite when $c=-1$. Similarly, the coefficient of x^{c+3} equated to zero implies

$$\text{therefore} \quad a_3 = \frac{2a_2}{(c+4)(c+2)} \tag{87}$$

The recurrence relation as obtained from Eq.(114) by demanding that the coefficient of x^{m+c} be zero.

$$(m+c+1)(m+c-1)a_m - 2a_{m-1} = 0 \tag{88}$$

$$a_m = \frac{2a_{m-1}}{(m+c+1)(m+c-1)} \tag{89}$$

Thus a_3 and all the subsequent coefficients are proportional to a_2 and hence becomes infinite, for $c = -1$, due to presence of a factor $(c+1)$ in the denominator of a_2, see Eq.(122). Since the overall constant a_0 arbitrary, we may select $a_0 = k(c+1)$ making a_2 and all the subsequent coefficients finite for both the values of $c = \pm 1$. With the choice $a_0 = k(c+1)$ and using the recurrence relation Eq.(125) in Eq.(117) one gets

$$Ly(x,c) \quad = \quad a_0(c^2-1)x^c = k(c-1)(c+1)^2x^c \tag{90}$$

It is apparent from the above equation that for $c = -1$ we have two linearly independent solutions given by

$$y_1(x) = y(x,c)\big|_{c=-1} \text{ and } y_2(x) = \left.\frac{dy(x,c)}{dc}\right|_{c=-1} \tag{91}$$

It can be explicitly checked that yet another solution, obtained from $y(x,c)$ by setting $c=1$, is proportional to the solution $y_1(x)$.

We shall now get explicit form of the two solutions Eq.(127), Eq.(120)Eq.(122),Eq.(123) and Eq.(125) give

$$a_1 = \frac{2a_0}{(c)(c+2)}, \qquad a_2 = \frac{2.2a_0}{(c+3)(c+2)(c+1)c} \tag{92}$$

$$a_3 = \frac{2^3 a_0}{(c+4)(c+3)(c+2)(c+2)(c+1)c} \tag{93}$$

and in general

$$a_m = \frac{2^m a_0}{(m+c+1)(m+c)\cdots(c+2)(m+c-1)(m+c-2)\cdots c} \tag{94}$$

Multiplying and dividing Eq.(130) by $[\Gamma(c+2)]^2$, the expression for a_m can be easily cast in the form

$$a_m = \frac{2^m a_0 \Gamma(c+2)\Gamma(c+2)}{\Gamma(m+c+2)\Gamma(m+c)c(c+1)} \tag{95}$$

Writing the series for $y(x,c)$, using Eq.(128),Eq.(129) and Eq.(131), we obtain

$$\begin{aligned} y(x,c) = a_0 x^c \Bigg\{ 1 + \frac{2x}{(c)(c+2)} + \frac{2^2\, x^2}{(c+1)c(c+3)(c+2)} + \cdots \\ + \frac{2^m \Gamma(c+2)\Gamma(c+2)}{\Gamma(m+c+2)\Gamma(m+c)c(c+1)} x^m + \cdots \Bigg\} \end{aligned} \tag{96}$$

Next we use $a_0 = k(c+1)$ and rewrite the above series as

$$\begin{aligned} y(x,c) = kx^c \Bigg\{ (c+1) + (c+1) \times \frac{2x}{(c)(c+2)} + + \frac{2^2\, x^2}{c(c+3)(c+2)} + \cdots \\ + \frac{2^m \Gamma(c+2)\Gamma(c+2)}{\Gamma(m+c+2)\Gamma(m+c)c} x^m + \cdots \Bigg\} \end{aligned} \tag{97}$$

One solution is obtained by setting $c = -1$ in Eq.(133), which apart from an overall constant can be written as

$$y_1(x) = \sum_{m=2}^{\infty} \frac{2^m}{(m!)(m-2)!} x^{m-1} = 2 \sum_{m=0}^{\infty} \frac{2^{m+1}}{(m!)(m-2)!} x^{m+1} \tag{98}$$

To obtain the second solution we differentiate Eq.(133) w.r.t c and set $c=-1$. This gives

$$\frac{dy(x,c)}{dc} \tag{99}$$

$$\begin{aligned} = \; & k \log x \, x^c \Bigg\{ (c+1) + (c+1)\frac{2x}{c(c+2)} + \frac{2^2\, x^2}{c(c+3)(c+2)} + \cdots \\ & \cdots + \frac{2^m \Gamma(c+2)\Gamma(c+2)}{\Gamma(m+c+2)\Gamma(m+c)c)} x^m + \cdots \Bigg\} \\ & + x^c \Bigg\{ 1 - \frac{2x}{c(c+2)} + \frac{\mathrm{d}}{\mathrm{d}c} \frac{2^2\, x^2}{c(c+3)(c+2)} + \cdots + \frac{\mathrm{d}}{\mathrm{d}c} \frac{2^m \Gamma(c+2)\Gamma(c+2)}{\Gamma(m+c+2)\Gamma(m+c)c)} x^m + \cdots \Bigg\} \end{aligned} \tag{100}$$

Computing the derivative of $\log a_m$, with a_m as in Eq.(131) we get

$$\log a_m = \log 2^m k + 2\log\Gamma(c+2) - \log\Gamma(m+c+2) - \log\Gamma(m+c) - \log c \tag{101}$$

Thus we have and setting $c = -1$ one gets

$$\begin{aligned} \frac{1}{a_m}\frac{da_m}{dc}\bigg|_{c=-1} &= \left\{2\psi(c+2) - \psi(c+m+2) - \psi(m+c) - \frac{1}{c}\right\}\bigg|_{c=-1} \\ &= -2\gamma - \psi(m+1) - \psi(m-1) + 1 \end{aligned} \tag{102}$$

Writing

$$\phi(n) = 1 + \frac{1}{2} + \frac{1}{3} = \cdots + \frac{1}{n} \tag{103}$$

and defining $\phi(0) = 0$, and using

$$\psi(n) = -\gamma + \phi(n-1), \text{ for } n > 1. \tag{104}$$

Eq.(102) can be simplified, for $m > 2$ to give

$$\frac{da_m}{dc}\bigg|_{c=-1} = a_m\left[\phi(m-2) + \phi(m) - 1\right] \tag{105}$$

$$= \frac{2^m\left[\phi(m-2) + \phi(m) - 1\right]}{m!\,(m-2)!} \tag{106}$$

Substituting back in Eq.(100) gives the series for the second solution as

$$y_2(x) = -2y_1(x)\log x + x^{-1}\Delta(x) \tag{107}$$

where $\Delta(x)$ is the series given by

$$\begin{aligned} \Delta(x) &= 1 - 2x + x^2 + \cdots + \frac{2^m\left[\phi(m) + \phi(m-2) - 1\right]}{m!\,(m-2)!}x^m + \cdots \\ &= \left\{1 - 2x + x^2 + \cdots + \sum_{m=3}^{\infty}\frac{2^m\left[\phi(m) + \phi(m-2) - 1\right]}{m!\,(m-2)!}x^{m+1} + \cdots\right\} \end{aligned}$$

The series solution obtained by setting $c = 1$ in $y(x,c)$ of Eq.(133)is proportional to $y_1(x)$.

§5 Case-IV :: Roots Differ by an Integer and Some Coefficient is Indeterminate

In this lecture we shall take up solution of an ordinary differential equation by the method of series solution. The example to be discussed is such that the difference of the roots of the indicial equation is an integer and some coefficient becomes indeterminate.

Consider the differential equation

$$\frac{d^2y}{dx^2} + x^2 y = 0 \tag{108}$$

Substituting

$$y = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{109}$$

in Eq.(135) we get

$$\sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} + x^2 \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{110}$$

or,

$$\sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} + \sum_{n=0}^{\infty} a_n x^{n+c+2} = 0 \tag{111}$$

The lowest power of x in the right hand side of Eq.(111) is x^{c-2}. This gives

$$a_0 c(c-1) = 0 \tag{112}$$

Therefore the two values of c are $c = 0$ and $c = 1$. Equating the coefficients of $x^{c-1}, x^c, x^{c+1}, x^{c+2}, \ldots$ to zero successively gives

$$a_1 c(c+1) = 0 \tag{113}$$

$$a_2 (c+1)(c+2) = 0 \tag{114}$$

$$a_3 (c+2)(c+3) = 0 \tag{115}$$

$$a_4 (c+4)(c+3) + a_0 = 0 \tag{116}$$

The recurrence relation obtained by considering the coefficient of x^{m+c+2} is

$$a_{m+4}(c+m+4)(c+m+3) + a_m = 0 \tag{117}$$

The solution for $c = 1$ can be constructed as before. For the case $c = 0$, however, we get fromEq.(113)

$$a_1 .0 = 0 \tag{118}$$

Thus a_1 cannot be fixed and is indeterminate. In this case we proceed as before except that we retain both a_0 and a_1 as unknown parameters.

Case $c = 0$:

Substituting $c = 0$ from Eq.(113) to Eq.(117) we get

$$a_2 = a_3 = 0; \quad a_4 = -\frac{a_0}{4.3} \tag{119}$$

$$a_{m+4} = -\frac{a_m}{(m+4)(m+3)} \tag{120}$$

Combining Eq.(119) and Eq.(120) we see that

$$a_2 = a_6 = a_1 0 \cdots 0 \tag{121}$$

and

$$a_3 = a_7 = a_1 1 \cdots 0 \tag{122}$$

Also

$$a_4 = -\frac{1}{4.3} a_0; \quad a_8 = -\frac{1}{8.7} a_4; \quad a_{12} = -\frac{1}{12.11} a_8 \tag{123}$$

$$a_5 = -\frac{1}{5.4} a_1; \quad a_9 = -\frac{1}{9.8} a_5; \quad a_{13} = -\frac{1}{13.12} a_9 \tag{124}$$

Solving Eq.(123) and Eq.(124) we get

$$a_4 = -\frac{1}{4.3} a_0; \quad a_8 = -\frac{1}{8.7.4.3} a_0; \quad a_{12} = -\frac{1}{12.11.8.7.4.3} a_0 \tag{125}$$

$$a_5 = -\frac{1}{5.4} a_1; \quad a_9 = -\frac{1}{9.8.5.4} a_1; \quad a_{13} = -\frac{1}{13.12.9.8.5.4} a_1 \tag{126}$$

The series solution in this case contains two parameters, which are not determined by the recurrence relations, and is given by

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \tag{127}$$

$$y_1(x) = 1 - \frac{x^4}{3.4} + \frac{x^8}{3.4.7.8} - \frac{x^{12}}{3.4.7.8.11.12} + \cdots \tag{128}$$

$$y_2(x) = x \left\{ 1 - \frac{x^4}{4.5} + \frac{x^8}{4.5.8.9} - \frac{x^{12}}{4.5.8.9.12.13} + \cdots \right\} \tag{129}$$

These two functions $y_1(x)$ and $y_2(x)$ represent two linearly independent solutions. What happens when one tries to construct the solution for the second value of c ? In this case we recover one of the above two solutions already obtained. This will now be demonstrated explicitly.

Case $c = 1$:

In this case we get

$$a_1 = a_2 = a_3 = 0 \tag{130}$$

$$a_{m+4} = -\frac{a_m}{(m+5)(m+4)} \tag{131}$$

We therefore get

$$a_4 = -\frac{1}{5.4} a_0; \quad a_8 = -\frac{1}{9.8} a_4; \quad a_{12} = -\frac{1}{13.12} a_8 \tag{132}$$

Compare the equations Eq.(132) with Eq.(124) . We now construct the series

$$y = x^c \sum a_n x^n \tag{133}$$

and get

$$y_2(x) = a_0 x \left\{ 1 - \frac{x^4}{4.5} + \frac{x^8}{4.5.8.9} - \frac{x^{12}}{4.5.8.9.12.13} + \cdots \right\} \tag{134}$$

This solution coincides with $y_2(x)$ of Eq.(129) except for an overall constant. Hence the most general solution of the differential equation Eq.(135) is given by Eq.(127)

§6 Point at Infinity

For a second order linear differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{135}$$

sometimes instead of a series solution in powers of x, it may be useful to expand in negative powers of x:

$$y(x,c) = x^c \sum_{n=0}^{\infty} a_n x^{-n} \tag{136}$$

This results on convergence etc. of this type of solutions are conveniently obtained by changing the independent variable from x to $t = 1/x$. The differential equation Eq.(135) written in terms of t becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \tilde{p}(t)\frac{\mathrm{d}y}{\mathrm{d}t} + \tilde{q}(t)y = 0 \tag{137}$$

where

$$\tilde{p}(t) = \frac{2}{t} - \frac{1}{t^2}p(t); \tilde{q}(t) = \frac{1}{t^4}q(1/t) \tag{138}$$

The behaviour of the series solution at $t = 0$ gives the answer for the behaviour of the solution in the inverse powers of x.