QFT-03 Lecture Notes

Space Time Transformations

Lorentz Group

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The Lorentz group is the set of all linear transformations on $x^\mu = (x^0 = t, \vec{x})$ such that $t^2 - \vec{x}^2$ is invariant. Writing the components of x^μ in a column

$$\tilde{X} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{1}$$

we write $x^{02} - \vec{x}^{\,2}$ as $\widetilde{X}^T \tilde{g} \widetilde{X}$, where $\widetilde{X}^T$ is transpose of the column vector $\widetilde{X}$ and $\tilde{g}$ is four by four matrix

$$\tilde{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2}$$

The metric tensor $g_{\mu\nu}$ defined by the matrix $\tilde{g}$. As usual the metric tensor will be used to raise and lower the Lorentz indices. We shall also use $x^2 = x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu$ to denote the invariant $x^2 \equiv x^{02} - \vec{x}^{\,2}$.

§1 Properties of Lorentz transformations

A Lorentz transformation is a linear transformation on $\tilde{X}$. Hence it can be represented by a 4×4 matrix $\widetilde{\Lambda}$

$$\widetilde{X}' = \widetilde{\Lambda}\widetilde{X}. \tag{3}$$

The condition that $\tilde{X}^T g \tilde{X}$ be invariant can be written as

$$\tilde{X}^T g \tilde{X} = \tilde{X}'^T g \tilde{X}' \tag{4}$$

and this requirement implies that $\widetilde{\Lambda}$ must satisfy

$$\widetilde{\Lambda}^T \tilde{g} \widetilde{\Lambda} = \tilde{g} \qquad \text{VerifyNow}. \tag{5}$$

An immediate consequence of (5) is obtained by taking its determinant and noting that $\det g = -1$. This gives

$$\begin{aligned} & (\det \widetilde{\Lambda}^T) \det(\tilde{g})(\det \widetilde{\Lambda}) = -1, \\ \text{or,} \qquad & (\det \widetilde{\Lambda})^2 = 1, \\ \text{or,} \qquad & \det \widetilde{\Lambda} = \pm 1. \end{aligned}$$

Also taking 00 component of (4), we get

$$\begin{aligned} & \widetilde{\Lambda}_{00}^2 - \widetilde{\Lambda}_{10}^2 - \widetilde{\Lambda}_{20}^2 - \widetilde{\Lambda}_{30}^2 = 1, \\ \text{or} \qquad & \widetilde{\Lambda}_{00}^2 = 1 + \widetilde{\Lambda}_{10}^2 + \widetilde{\Lambda}_{20}^2 + \widetilde{\Lambda}_{30}^2 \geq 1. \end{aligned} \tag{6}$$

§2 Special Cases

We shall briefly describe some simple transfromations that leave $x^\mu x_\mu$ invariant. All these transformations are elements of the Lorentz group.

1. Time reversal: $x^0\prime = -x^0, \vec{x}\,' = \vec{x}$ and

$$\widetilde{\Lambda} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{7}$$

$$\det \widetilde{\Lambda} = -1, \qquad \widetilde{\Lambda}_{00} = -1. \tag{8}$$

2. Parity: $x^{0\prime} = x^0, \vec{x}\,' = -\vec{x}$ and

$$\widetilde{\Lambda} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{9}$$

$$\det \widetilde{\Lambda} = -1, \qquad \widetilde{\Lambda}_{00} = 1. \tag{10}$$

3. Spatial rotations: $x^{0\prime} = x^0$, and $\vec{x}\,' = R\vec{x}$ and R is such that $\vec{x}\,' \cdot \vec{x}\,' = \vec{x} \cdot \vec{x}$. The matrix $\widetilde{\Lambda}$ for rotations is of the form

$$\widetilde{\Lambda} = \begin{pmatrix} 1 & | & 0 & & \\ - & | & - & - & - \\ 0 & | & R & & \end{pmatrix} \tag{11}$$

where 3×3 matrix R satifies

$$R^T R = I. \tag{12}$$

This equation implies

$$\det R = \pm 1. \tag{13}$$

Rotations corresponding to $\det R = 1$ are called *proper rotations*. A proper rotation can be specified by giving the axis of rotation and angle of rotation.

An *improper rotation*, *i.e.* with $\det R = -1$, can be written as a proper rotation followed by a parity transformation.

Proper rotations preserve the handedness of the coordinate sysytem while under an improper rotation right handed coordinate system becomes left handed and and vice versa.

4. Lorentz boosts: The Lorentz boosts correspond to space time transformations from a frame to another frame K' moving with uniform velocity in a given direction. Simple boosts correspond to the situations when the relative velicity is along one of the coordinate axes. Thus we have the transformation equations

$$x^{3\prime} = \frac{x^3 - vt}{\sqrt{1-\beta^2)}}, \qquad t' = \frac{t - vx^3/c^2}{\sqrt{1-\beta^2}}, \qquad \beta = v/c. \tag{14}$$

for a transformation along the x^3 axis. For these transformations

$$\det \widetilde{\Lambda} = 1, \qquad \widetilde{\Lambda}_{00} = 1. \tag{15}$$

§3 Improper transformations

Combination of one of the discrete transformations, time reversal or partiy, with proper rotations (or propper Lorentz boosts with $\det \widetilde{\Lambda} > 0$) give improper transformations. So
Parity and rotations $\longrightarrow$ improper rotations with $\det R = -1$
Time reversal or Parity and Lorentz boost $\longrightarrow$ improper boosts with $\det \widetilde{\Lambda} = -1$.
The set of all Lorentz transformations can be divided into following susbets.

(i) $\det \widetilde{\Lambda} = +1, \widetilde{\Lambda}_{00} \geq 1$

(i) $\det \widetilde{\Lambda} = +1, \widetilde{\Lambda}_{00} \leq -1$

(i) $\det \widetilde{\Lambda} = -1, \widetilde{\Lambda}_{00} \geq 1$

(i) $\det \widetilde{\Lambda} = -1, \widetilde{\Lambda}_{00} \leq -1$

These subsets are disconnected and starting from one of the above four subsets one cannot reach another element of a different set by a continuous transformation. This is because under a continuous change of elements of the matrix $\widetilde{\Lambda}$,the values of $\det \widetilde{\Lambda}$ as well as of $\widetilde{\Lambda}_{00}$ cannot change form 1 to -1. Of these subsets, only (i) is a subgroup; (ii) to (iv) do not have the identity element and are not subgroups. It can be proved that the Lorentz group consists of only these four disconnected subsets.

§4 Some notation

The matrix equation (3), in the four vector notation, will be written as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \tag{16}$$

The $(\mu\nu)$ element of the matrix $\widetilde{\Lambda}$, $\widetilde{\Lambda}_{\mu\nu}$, will be written as $\Lambda^\mu{}_\nu$; here the first index μ is the row index and the second index ν is the column index.

The transpose of matrix $\widetilde{\Lambda}$, $\widetilde{\Lambda}^T$, will be obtained by interchanging row and column indices and we wrtie $\mu\nu$ element of $\widetilde{\Lambda}^T$ as $(\widetilde{\Lambda^T})_{\mu\nu} = (\widetilde{\Lambda})_{\nu\mu} = \Lambda^\nu{}_\mu$.

Next note that (5) implies that

$$g\widetilde{\Lambda}^T g\widetilde{\Lambda} = I \tag{17}$$

This shows that the matrix $g\widetilde{\Lambda}^T g$ is inverse of the matrix $\widetilde{\Lambda}$

$$\widetilde{\Lambda}^{-1} = g\widetilde{\Lambda}^T g. \tag{18}$$

The inverse relation

$$\widetilde{X} = \widetilde{\Lambda}^{-1}\widetilde{X}\,'. \tag{19}$$

will be written as

$$\begin{aligned} x^\mu &= (g\widetilde{\Lambda}^T g)^\mu{}_\nu x'^\nu \\ &= g^{\mu\alpha}\Lambda^\beta{}_\alpha g_{\beta\nu} x'^\nu. \end{aligned}$$

We write this relation as

$$\boxed{x^\mu = \Lambda_\nu{}^\mu x'^\nu} \tag{20}$$

where we have defined

$$g^{\mu\alpha}\Lambda^\beta{}_\alpha g_{\beta\nu} \equiv \Lambda_\nu{}^\mu. \tag{21}$$

In order to read the above equation as a matric equation, use symmetry of g to rewrite the above relation as

$$\Lambda_\nu{}^\mu = g_{\nu\beta}\Lambda^\beta{}_\alpha g^{\alpha\mu}. \tag{22}$$

The equation (20) should be interpreted as lowering the row index β to ν and raising the column index α to μ. In this process of raising and lowering the indices, a column index should remain column index and a row index should remain row index.

Finally we note that the relation (17) can be written aa

$$\Lambda^\mu{}_\alpha\Lambda_\mu{}^\beta = \delta_\alpha^\beta = \Lambda^\beta{}_\nu\Lambda^\nu{}_\alpha \tag{23}$$

§5 Vectors, Tensors and Fields

Contravariant vectors Any four component object $a_\mu, \mu = 0,1,2,3$, such that its components transform under a Lorentz transformation $\widetilde{\Lambda}$ like components of x^μ, as in (3), is called a *contra variant vector*:

$$a^\mu = \Lambda^\mu{}_\nu a^\nu \tag{24}$$

Scalar and vector fields Under a Lorentz transformation given by $\widetilde{\Lambda}$, (3), a *scalar field* is a function $\phi(x)$ which transforms as

$$\phi(x) \xrightarrow{\Lambda} \phi'(x') = \phi(x). \tag{25}$$

A four component field $\phi^\mu(x), \mu = 0, 1, 2, 3$ is called a *vector field* if under Lorentz transformation $\widetilde{\Lambda}$ its components transforms as

$$\phi^\mu(x) \xrightarrow{\Lambda} \phi'^\mu(x') = \Lambda^\mu{}_\nu \phi^\nu(x). \tag{26}$$

Covariant vectors Consider a scalar field $\phi(x)$ and let us work out the transformation properties of its four derivative $\frac{\partial\phi(x)}{\partial x^\mu}$ under Lorentz transformtions. Using

$$\frac{\partial}{\partial x^\mu} \longrightarrow \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \tag{27}$$

we will have the derivative transforming as

$$\frac{\partial\phi(x)}{\partial x^\mu} \longrightarrow \frac{\partial\phi'(x')}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial\phi(x)}{\partial x^\nu} \qquad \boxed{\text{Use (25)}} \tag{28}$$

Differentiating relation (20), w.r.t. x'^α and using $\dfrac{\partial x'^\nu}{\partial x'^\alpha} = \delta^\nu_\alpha$ in the r.h.s. we get

$$\frac{\partial x^\mu}{\partial x'^\alpha} = \Lambda^\mu{}_\nu \frac{\partial x'^\nu}{\partial x'^\alpha} = \Lambda^\mu{}_\alpha \tag{29}$$

and hence $\frac{\partial\phi}{\partial x^\mu}$ transforms as

$$\frac{\partial\phi}{\partial x^\mu} \xrightarrow{\Lambda} \frac{\partial\phi'(x')}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial\phi(x)}{\partial x^\nu} = \Lambda^\nu{}_\mu \frac{\partial\phi(x)}{\partial x^\nu}. \tag{30}$$

The above equation is also be written as

$$\partial_\mu\phi(x) \xrightarrow{\Lambda} \partial'_\mu\phi'(x') = \Lambda^\nu{}_\mu \partial_\nu\phi(x). \tag{31}$$

A four vector with components transforming in the above manner, *i.e.*

$$A_\mu \longrightarrow A'_\mu = \Lambda^\nu{}_\mu A_\nu, \tag{32}$$

is called a *covariant vector.*

Tensors, and tensor fields, of higher rank are objects with more two or more Lorentz indices with each index transforming as in (20). It is straightforward to check that contraction of a covariant and a contra variant indices leads to Lorentz scalar.