Course of Lectures in Mechanics:: Unit-A[1]

Some Mathematical Preparation

(Printer Freindly Version)

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Unit Overview

Syllabus

Vector algebra Preliminaries: Kronecker delta and LeviCivita symbol; Einstein summation convention. 123 notation for vector components. Dot and cross products of two vectors and vector identities.

Prerequisites

Vector algebra; dot and cross products; Triple product; Geometrical representation; Components of a vector in a system of coordinate axes; Vector algebra identities; Direction cosines; Matrix algebra; Determinants and their properties.

Lessons

1. A Quick Review of Vector Algebra

2. Summation Convention, ϵ, δ symbols and All That

References

Following is an incomplete list of reference for reading on introductory and specialized range of topics in matrices and vector analysis.

1. Shanti Narayan, *A Text Book of Matrices*, Revised by P. K. Mittal, S. Chand and Co, Delhi (2010)

2. Andrilli S. and Hecker D., *Elementary Linear Algebra*, Academic Press, California USA Fourth edition (2010)

3. Bellman R.,*Introduction to Matrix Analysis*, Textbook Publishers (2003)

4. Khan Academy :: Introduction to matrices
   `https://www.khanacademy.org/math/algebra-home/alg-matrices`

5. Emma Thomas, Matrices and Determinants
   `http://www.maths.surrey.ac.uk/explore/emmaspages/option1.html`
   A page developed for The University of Surrey, England.

6. Murphy G. M. and Margenau H. *The Mathematics of Physics and Chemistry*, Van Nostrand (1967).

7. Spain B., *Vector Analysis*, Van Nostrand Reinhold (1967).

Lesson A-1::A Quick Review of Vectors

§0.1 Lesson Overview

Learning Goals

In this lesson, we shall begin with vectors as geometrical objects. A quick review of a few vector algebra identities will be presented. With a choice of coordinate system, vectors are described as objects with three components. We will present a result on change in components of a vector when coordinate axes are changed.

Prerequisites

A first exposure to vector algebra; Dot, cross and triple products. Components of a vector along coordinate axes.

§0.2 Vectors as Geometrical Objects

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The vectors are introduced geometrical objects having a magnitude and direction. Then one can define various operations on vectors. These include multiplication by a real number, addition of two vectors, taking dot and cross products of two vectors.

A large variety of physical quantities, such as displacement, velocity etc., appear as vectors. The laws of physics are formulated as vector (tensor equation) equations. In order to be able to make numerical predictions and to compare them with experimental data, geometric description of vector physical quantities turns out inadequate,if not useless. While the orbit of a planet around the Sun can be geometrically described as ellipses, but to use laws of physics to make predictions and detailed numerical comparisons observations it is essential to introduce a coordinate system and work with the three components of the position vector.

Notation & Convention:
We shall use boldface letters, $\mathbf{A}, \mathbf{B}, \mathbf{C}$.., to denote vectors.
If $\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vectors along the coordinate axes, a given vector can be expressed as a linear combination of these unit vectors along the three axes.

$$\mathbf{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}. \tag{1}$$

Here A_x, A_y, A_z are the components of vectors in chosen set of axes. We will use the notation $\vec{A} = (A_x, A_y, A_z)$ to denote the set of the three components of a vector.
Also, we shall use $\widetilde{\mathsf{A}}$ to denote the the column vector of components of a vector $\mathbf{A}$.

$$\widetilde{\mathsf{A}} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}. \tag{2}$$

We will also use notation $\vec{R} = (x, y, z)$ for the components of vector. Frequently the components will be assembled in a column vector to write

$$\widetilde{\mathsf{R}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3}$$

§0.3 An Example

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Why This Example? To illustrate use of different notations for vectors.

Let K' be a set of axes obtained by carrying out a rotation by and angle α on a set of coordinate axes K. Find relation between components of position vector of a point w.r.t. the two sets K and K'.

Let $\mathbf{R}$ denote the position vector of a point P. The notation for the components along the axes in K and K' will be written as

$$\vec{R} = (x, y, z), \qquad \vec{R}' = (x', y', z') \tag{4}$$

and

$$\widetilde{R} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; \qquad \widetilde{R}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \tag{5}$$

Since the rotation is performed about the Z axis, the Z' axis coincides with Z axis and we have $z' = z$.

However components of $\mathbf{R}$, and also the unit vectors, along X, Y and along X', Y' axes, will be different.

We use the notation $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to denote the unit vectors along K axes. The unit vectors along the K' axes will be denoted by $\mathbf{i}', \mathbf{j}', \mathbf{k}'$, with $\mathbf{k}' = \mathbf{k}$. Thus vector $\mathbf{R}$ can be written in two ways as

$$\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{6}$$

$$\mathbf{R} = x'\mathbf{i}' + y\mathbf{j}' + z\mathbf{k}. \tag{7}$$

Components of $\mathbf{R}$ w.r.t. K' are obtained by taking its dot product with $\mathbf{i}'$ and $\mathbf{j}'$. Thus, from (6) we get

$$x' = \mathbf{R} \cdot \mathbf{i}' = x(\mathbf{i} \cdot \mathbf{i}') + y(\mathbf{j} \cdot \mathbf{i}') \tag{8}$$

$$y' = \mathbf{R} \cdot \mathbf{j}' = x(\mathbf{i} \cdot \mathbf{j}') + y(\mathbf{j} \cdot \mathbf{j}'). \tag{9}$$

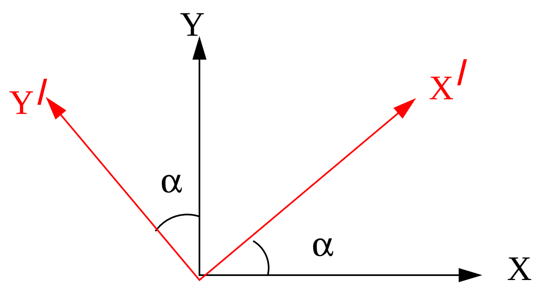

Fig. **1** Rotation about Z axis

Various dot products can be read in terms of the angle of rotation α from Fig.1 and we get

$$\mathbf{i} \cdot \mathbf{i}' = \cos\alpha, \qquad \mathbf{j} \cdot \mathbf{i}' = \sin\alpha \tag{10}$$

$$\mathbf{i} \cdot \mathbf{j}' = -\sin\alpha, \qquad \mathbf{j} \cdot \mathbf{j}' = \cos\alpha. \tag{11}$$

Substituting the above expressions in Eq.(8)-(9), we get

$$x' = x\cos\alpha + y\sin\alpha \tag{12}$$

$$y' = -x\sin\alpha + y\cos\alpha \tag{13}$$

$$z' = z. \tag{14}$$

and of course $z' = z$. In the matrix notation we writethis relationship as

$$\widetilde{R}' = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \widetilde{R}. \tag{15}$$

§0.4 Vector Algebra Identities

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The dot and cross product satisfy the following identities.

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \quad (16)$$

$$|\vec{A} \times \vec{B}|^2 + (\vec{A} \cdot \vec{B})^2 = \|A\|^2 \|B\|^2 \quad (17)$$

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{B} \cdot \vec{C})(\vec{A} \cdot \vec{D}) \quad (18)$$

$$\left[\vec{A} \times \vec{B}, \vec{B} \times \vec{C}, \vec{C} \times \vec{A}\right] = \left[\vec{A}, \vec{B}, \vec{C}\right]^2 \quad (19)$$

And some more

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0 \quad (20)$$

$$[\vec{A}, \vec{B}, \vec{C}]\vec{D} = (\vec{A} \cdot \vec{D})(\vec{B} \times \vec{C}) + (\vec{B} \cdot \vec{D})(\vec{C} \times \vec{A}) + (\vec{C} \cdot \vec{D})(\vec{A} \times \vec{B}) \quad (21)$$

If $[\vec{A}, \vec{B}, \vec{C}] \neq 0$, every vector $\vec{X}$ can be represented as

$$\vec{X} = \alpha\vec{A} + \beta\vec{B} + \gamma\vec{C} \quad (22)$$

where

$$\alpha = \frac{\vec{X} \cdot \vec{B} \times \vec{C}}{[\vec{A}, \vec{B}, \vec{C}]}; \quad \beta = \frac{\vec{X} \cdot \vec{C} \times \vec{A}}{[\vec{A}, \vec{B}, \vec{C}]}; \quad \gamma = \frac{\vec{X} \cdot \vec{A} \times \vec{D}}{[\vec{A}, \vec{B}, \vec{C}]} \quad (23)$$

The area of a parallelogram with sides represented by the vectors $\vec{A}, \vec{B}$ is given by $||\vec{A} \times \vec{B}||$ and we have

$$|\vec{A} \times \vec{B}|^2 = \begin{vmatrix} \vec{A} \cdot \vec{A} & \vec{A} \cdot \vec{B} \\ \vec{B} \cdot \vec{A} & \vec{B} \cdot \vec{B} \end{vmatrix} \quad (24)$$

The volume V of a parallelopiped with represented by the vectors $\vec{A}, \vec{B}$ and $\vec{C}$ is given by

$$V^2 = \begin{vmatrix} \vec{A} \cdot \vec{A} & \vec{A} \cdot \vec{B} & \vec{A} \cdot \vec{C} \\ \vec{B} \cdot \vec{A} & \vec{B} \cdot \vec{B} & \vec{B} \cdot \vec{C} \\ \vec{C} \cdot \vec{A} & \vec{C} \cdot \vec{B} & \vec{C} \cdot \vec{C} \end{vmatrix} \quad (25)$$

References Wikipedia , *Vector Algebra Relations*
https://en.wikipedia.org/wiki/Vector_algebra_relations

§0.5 Change of coordinate axes

We know that every vector **A** can be expressed as linear combination of unit vectors along the three coordinate axes. The coefficients are called components of the vector. The components of a given vector will be different w.r.t. different coordinate systems. Here we present a way of relating the components of a vector w.r.t. two different sets of axes.

Let us assume that we have two right handed coordinate systems, K, K', whose origins coincide but the axes are orientated differently. The coordinates of a point P as seen in two frames will be different. We wish to find relation between the components w.r.t. the two sets of axes.

Define direction cosines

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote unit vectors along the axes K and $\mathbf{l}, \mathbf{m}, \mathbf{n}$ denote unit vectors along the new axes. Let the components of unit vectors $\mathbf{l}, \mathbf{m}, \mathbf{n}$ w.r.t. the old axes be written as

$$\vec{\ell} = (\ell_1, \ell_2, \ell_3); \quad \vec{m} = (m_1, m_2, m_3); \quad \vec{n} \;=\; (m_1, m_2, m_3). \tag{26}$$

Then we have

$$\mathbf{l} = \ell_1\mathbf{i} + \ell_2\mathbf{j} + \ell_3\mathbf{k}; \qquad \mathbf{m} = m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k}; \qquad \mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k} \tag{27}$$

Get a vector as linear combinations in the two bases

Let $(x, y, z) \equiv \vec{r}$ and $(x', y', z') \equiv \vec{r}'$ denote the components of the position vector $\overrightarrow{OP}$ of P the w.r.t. the frames K, K'. Thus we have

$$\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} && (28)\\ &= x'\mathbf{l} + y'\mathbf{m} + z'\mathbf{n} && (29) \end{aligned}$$

Using (27) we get

$$\begin{aligned} \mathbf{r} &= x'\mathbf{l} + y'\mathbf{m} + z'\mathbf{n} && (30)\\ &= x'(\ell_1\mathbf{i} + \ell_2\mathbf{j} + \ell_3\mathbf{k}) + y'(m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k}) + z'(n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}) && (31)\\ &= (\ell_1x' + m_1y' + n_1z')\mathbf{i} + (\ell_2x' + m_2y' + n_2z')\mathbf{j} + (\ell_3x' + m_3y' + n_3z')\mathbf{k} && (32) \end{aligned}$$

Comparing the last expression with $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we get

$$x = (\ell_1x' + m_1y' + n_1z'); \quad y = (m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k}); \quad z = (\ell_3x' + m_3y' + n_3z') \tag{33}$$

Thus we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \tag{34}$$

Get the rotation matrix

Using the fact that vectors $\mathbf{l}, \mathbf{m}, \mathbf{n}$ are pairwise orthogonal unit vectors, it is easy to see that the inverse relation is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{35}$$

We introduce the notation

$$R = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}, \tag{36}$$

and also a column vector notation for vectors,

$$\widetilde{\mathsf{r}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \widetilde{\mathsf{r}}\,' = \begin{pmatrix} x\,' \\ y\,' \\ z\,' \end{pmatrix}. \tag{37}$$

The transformation equation (35), relating the components of $\mathbf{r}$ w.r.t the coordinate frames $K, K\,'$, takes a compact form

$$\widetilde{\mathsf{r}}\,' = R\widetilde{\mathsf{r}} \tag{38}$$

The matrix R will be called *rotation matrix* for change of axes reference frame from $K, K\,'$.

Important:

- When working with *only one coordinate system* there is no need to distinguish between $\mathbf{A}$ and $\vec{A}$. These two can be used interchangeably.
- When working with two or more coordinate systems $K\,', K'', \ldots$, we use $\vec{A}\,', \vec{A}''$ to denotes components w.r.t systems $K\,', K'', \ldots$.
- The components w.r.t. different coordinate systems will be collectively written as

$$\vec{A}\,' = (A_x\,', A_y\,', A_z\,'); \quad \text{and} \quad \vec{A}'' = (A''_x, A''_y, A''_z) \tag{39}$$

- The vector itself can be written as

$$\begin{aligned} \mathbf{A} &= A\,'_x\hat{i}\,' + A\,'_y\hat{j} + A\,'_z\hat{k} \qquad (40) \\ &= A''_x\hat{i}'' + A''_y\hat{j}'' + A''_z\hat{k}'' \qquad (41) \end{aligned}$$

- Frequently, following matrix notation of assembling the components of a vector in a column vector turns out to be very convenient. A vector $\mathbf{A}$ in the matrix notation will be denoted by $\widetilde{\mathsf{A}}$, where

$$\widetilde{\mathsf{A}} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{42}$$

- Finally, the "$1-2-3$" notation for the components of a vector $\vec{A} = (A_1, A_2, A_3)$ will also be used in place of "`x-y-z`" notation $\vec{A} = (A_x, A_y, A_z)$.

Questions for you

1. *Verify that the transpose of matrix R equals the inverse of R, $R^T R = I$ and that* $\det R = 1$.

2. Using orthogonality property of the rotation matrix, show that the dot product of two vectors remains same when computed using the components in two different coordinate systems. This is expected as the values of length of a vector and angles between two vectors does not depend upon the choice of coordinate system.

Question for You

[1] Use orthogonality property of K' axes to argue that the transpose of matrix R is its inverse $R^T R = RR^T = I$. <<Click for a Hint>>
A matrix having the property is defined to be an *orthogonal matrix*. The above property of matrix R means that the rotation matrices are orthogonal matrices.

§0.6 EndNotes

1. For a quick review of vector algebra see Murphy[0] Ch4; Griffiths[0] Ch1; For use of vectors in Physics see Feynman Lectures Vol-I[0] Ch 11.

2. The matrix R relating components of a vector in two different coordinate systems is an orthogonal matrix, see Eqs.(34) - (36).

Properties of orthogonal matrices We summarize some important properties of $N \times N$ orthogonal matrices.

It is useful to know that the set of $N \times N$ orthogonal matrices obey the following properties.

1. If O_1, O_2 are orthogonal matrices, then the product $O_1 O_2$ is also N dimensional orthogonal matrix.

2. The multiplication of matrices being associative property, we have $O_1(O_2 O_3) = (O_1 O_2) O_3)$ for orthogonal matrices too.

3. The identity matrix is orthogonal matrix.

4. If O is an orthogonal matrix, its inverse matrix O^{-1} is also orthogonal.

The set of all $N \times N$ orthogonal matrices obey all the requirements for a *group*. This group is called *orthogonal group* $O(N)$in N dimensions. The set of all $N \times N$ orthogonal matrices with $\det O = 1$ is a group $SO(N)$, known as the *special orthogonal group* in N dimensions.

References Watch this video to get started on definition of groups.

Lesson A-2 :: Summation Convention and All That

§7 Learning Goals ⇑–§1–§2–§3–⇓

You will learn about Einstein summation convention, Kronecker delta symbol and Levi-Civita epsilon symbol. Examples of usage of Kronecker delta and Levi-Civita symbols to vector algebra are presented.

§8 Summation Convention ⇑–§1–§2–§3–⇓

Einstein Summation Convention

We describe the Einstein summation convention and give some examples.

1. **Summation convention**
   If $\vec{x} = (x_1, x_2, x_3)$ is vector, square of its length is given by
   $$|\vec{x}|^2 = \sum_{i=1}^{3} x_i^2.$$
   We can rewrite it as
   $$|\vec{x}|^2 = \sum_{i=1}^{3} x_i x_i.$$
   In this form the index i is repeated and is summed over all values. The *Einstein summation convention* says all repeated indices are automatically summed over all possible values. With this convention we write
   $$|\vec{x}|^2 = x_i x_i.$$

2. **Dummy index**
   The index which is summed over all values is called a *dummy index*. A dummy index can be replaced with any other index taking the same set of values. Thus we can write $|\vec{x}|^2$ as $x_i x_i$, or as $x_j x_j$. Obviously the two expressions are equal.

3. **Free index must balance**
   An index which appears only once in an expression is not summed, is called a *free index*. Every term of an equation (or an expression) the free indices must balance.

4. **A relation having having a free index**
   If an index appears as a free index in an equation, it is understood, by convention,

that the hold for all values of the free index. As an example, matrix multiplication of a column vector u by a matrix, $v = Au$, is normally written as

$$v_i = \sum_{j=1}^{N} A_{ij} u_j, i = 1, \ldots, N, \tag{43}$$

With the above convention we will write it as

$$v_i = A_{ij} u_j \tag{44}$$

In the above equation $i$is a free index. It is understood that the above equation holds for all values of the free index i.

Kronecker Delta and Levi-Civita Symbols

⇑

Convention In this write up we assume Einstein summation convention for repeated indices.

Definition 1 *The Kronecker delta symbol δ_{ij} is defined as*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{45}$$

Definition 2 *The Levi-Civita symbol ϵ_{ijk} (with three indices) is a completely anti-symmetric under exchange of any two indices. So for example*

$$\epsilon_{ijk} = -\epsilon_{jik}; \epsilon_{ijk} = -\epsilon_{ikj}; \epsilon_{kij} = -\epsilon_{kij}.$$

Here the indices ijk take values from 1 to 3.

The symbol ϵ_{ijk} has only one independent component and we have $\epsilon_{123} = 1$. All other components are related to ϵ_{123} and turn out to be either zero or ± 1.

The definition of the Levi-Civita is easily generalized to the case of any number of indices. So with N indices $i_1, i_2, ..., i_N$ all taking values $1, 2, ..., N$, we have the symbol $\epsilon_{i_1, i_2, \ldots i_N}$ antisymmetric under exchange of any pair of two indices and $\epsilon_{123..N} = 1$.

)|(**Short Examples 1** *We explicitly list values of Kronecker delta and epsilon symbols when the indices run from 1 to 3.*

(1a) $\delta_{11} = \delta_{22} = \delta_{33} = 1$

(1b) $\delta_{12} = \delta_{21} = \delta_{23} = \delta_{32} = \delta_{31} = \delta_{13} = 0$

(1c) The six non-zero components of epsilon symbol are

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$$

$$\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$$

(1d) All other components of ϵ_{ijk} vanish when any two indices coincide. So, for example

$$\epsilon_{111} = \epsilon_{222} = \epsilon_{333} = 0$$

$$\epsilon_{112} = \epsilon_{122} = \epsilon_{133} = \ldots = 0$$

A useful result *If f_{ijk} is any object which is totally antisymmetric in its indices, then it must be proportional to the Levi-Civita symbol. Thus*

$$f_{ijk} = C\epsilon_{ijk}; \text{ and } C = f_{123}$$

Some identities *We give some identities of Kronecker delta and the Levi-Civita symbols for the case when the indices take three values 1,2,3.*

$$\delta_{ii} = 3; \qquad \epsilon_{ijk}\epsilon_{ijk} = 6 \tag{46}$$

For the Levi-Civita symbol we have the following identities.

$$\epsilon_{ijk}\,\epsilon_{ljk} = 2\,\delta_{il} \tag{47}$$

$$\epsilon_{ijk}\,\epsilon_{lmk} = (\,\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}) \tag{48}$$

$$\epsilon_{ijk}\,\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{49}$$

The determinant of a 3×3 matrix X has an expression in terms of Levi-Civita symbol.

$$\det X = \frac{1}{3!}\epsilon_{ijk}\epsilon_{\ell mn}X_{i\ell}X_{jm}X_{kn}. \tag{50}$$

This result generalizes matrices having any dimension.

Examples

Summation convention

)|(**Short Examples 2** *Let S_{ij} and A_{ij} are respectively symmetric and anti-symmetric under exchange ij and T_{ij} be arbitrary second rank tensor. Then*

- *(2a)* $S_{ij}T_{ij} = \frac{1}{2}S_{ij}\Big(T_{ij} + T_{ji}\Big)$
- *(2b)* $A_{ij}T_{ij} = \frac{1}{2}A_{ij}\Big(T_{ij} - T_{ji}\Big)$
- *(2c)* $S_{ij}A_{ij} = 0$.

Proof of (2a) Let S_{ij} be symmetric under exchange of indices $i \leftrightarrow j$ and T_{ij} be arbitrary tensor of rank 2. Thus we are given $S_{ij} = S_{ji}$. We will now show that

$$S_{ij}T_{ij} = \frac{1}{2}\Big(S_{ij}(T_{ij} + T_{ji})\Big).$$

Let σ denote the left hand side, $\sigma = S_{ij}T_{ij}$ Consider

$$\sigma = S_{ij}T_{ij} = S_{ji}T_{ij} \qquad \text{used given symmetry property of S} \tag{51}$$

Now replace dummy indices i, j by a new set mn to get

$$\sigma = S_{ji}T_{ij} = S_{nm}T_{mn} \quad \text{replaced } i \to m, j \to n \tag{52}$$

$$= S_{ij}T_{ji} \quad \text{replaced } m \to j, n \to i \tag{53}$$

This implies that the $\frac{1}{2}\Big(S_{ij}\big(T_{ij}+Tji\big)\Big) = \frac{\sigma}{2}+\frac{\sigma}{2} = \sigma$. which is the desired result. Proof of (2b) is written along similar lines. For a proof of (2c), use (2a) or (2b).

Use of ϵ, δ symbols in vector algebra

The use of Kronecker delta and Levi-Civita epsilon symbols for vector algebra and vector calculus simplifies computations. Here we give a few elementary examples to illustrate usage of these symbols.

[1] The dot product of two vectors $\vec{A} \cdot \vec{B}$ can be written as

$$\vec{A} \cdot \vec{B} = \delta_{jk}A_jB_k \tag{54}$$

[2] The cross product of two vectors $\vec{C} = \vec{A} \times \vec{B}$ can be written as

$$C_i = \epsilon_{ijk}A_jB_k \tag{55}$$

[3] The triple product $[\vec{A}, \vec{B}, \vec{C}]$ can be represented as

$$[\vec{A}, \vec{B}, \vec{C}] = \epsilon_{ijk}A_iB_jC_k \tag{56}$$

[4] Using the above expression is is easy to see that the cross product of a vector with itself vanishes. This is seen as follows. Let $\vec{C} = \vec{A} \times \vec{A}$, then

$$C_i = \epsilon_{ijk}(A_jA_k). \tag{57}$$

Here ϵ_{ijk} is antisymmetric under exchange $j \leftrightarrow k$ whereas A_jA_k is symmetric. Hence the sum over all jk vanishes.

[5] Vector algebra identities can be used to derive identities for the Kronecker delta and Levi-Civita epsilon symbols. For example

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{B} \cdot \vec{C})(\vec{A} \cdot \vec{D})$$

implies

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn}. - \delta_{km}\delta_{jn}$$

The proof of this result is left as an exercise for the reader.

§9 EndNotes

⇑–§1–§2–§3–⇓

1. **Food for your thought**

   (a) Writing out all terms for a 2×2 matrix A, explicitly verify that

   $$\epsilon_{ij}\epsilon_{mn}A_{im}A_{jn} = 2\det A.$$

   where the indices i, j, m, n take values 1 and 2.

   (b) For a three by three matrix A show that

   $$\det A = \epsilon_{ijk}A_{i1}A_{j2}A_{k3}.$$

2. The discussion of Kronecker δ and summation convention presented here is based on Woodhouse [0] Examples 3.1-3.6.

3. For introduction to Kronecker delta and Levi Civita Symbol and applications to vector calculus and electromagnetic theory see, for example `https://arxiv.org/pdf/1406.3060.pdf`

4. 0space Link for Levi Civita tensor On 0space.org ;
   See also Kronecker Delta function δ_{ij} and Levi-Civita (Epsilon symbol ϵ_{ijk}

Bibliography