ME-08 Lessons

EOM in Non Inertial Frames

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§1 Lesson Overview

Prerequisites

Newton's laws of motion; Some elementary ideas about inertial and non-inertial frames.

Lesson Objectives

To relate accelerations of a body as seen by an observer in an inertial frame and a non inertial frame. There are two cases of our present interest. These are as follows.

1. To obtain equation of motion in a linearly accelerated frame;
2. To obtain equation of motion in a rotating frame to obtain expressions for centrifugal and Coriolis forces.

§2 EOM in a Frame with Constant Acceleration

Basically there are two types of non inertial frames that are of interest.

- Frames accelerating with a constant acceleration;
- Rotating frames.

The Newton's Laws are not valid in non inertial frames. I cannot write

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

in a non inertial frame. So the question is how do write the EOM in accelerating and rotating frames? The E OM will look different and what are new consequences of Newton's Laws being different in non inertial frames?

The first case is simple. The examples are that of an lift, car or a train accelerating with a constant acceleration. So how do get the EOM.

I will always K for an inertial frame. In most cases it will clear which frame is inertial. If it is not, we must identify an inertial frame. I assume that there is an observer O who is performing an experiment and taking data. Then we have another frame $K\,'$ accelerating with a constant acceleration. Let there be an observer $O\,'$ in the frame $K\,'$.

Let $\vec{x}(t)$ and $\vec{x}\,'(t)$ be the position vectors of a point at time t as seen by the two observers O and $O\,'$ respectively. Let the positions of the axes in the two frames coincide at time $t = 0$. Also let $\vec{x}_O(t)$ be the position vector of the origin of K' at time t as seen by the observer O.

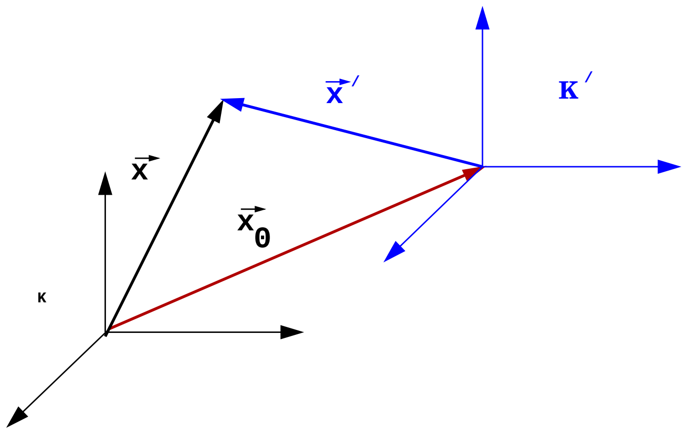

Fig. **1**

Then it is obvious that

$$\vec{x}(t) = \vec{x}_0(t) + \vec{x}\,'(t) \tag{1}$$

Differentiating the above equation two times we get

$$\frac{d\vec{x}(t)}{dt} = \frac{d\vec{x}_0(t)}{dt} + \frac{d\vec{x}'(t)}{dt} \tag{2}$$

$$\frac{d^2\vec{x}(t)}{dt^2} = \frac{d^2\vec{x}_0(t)}{dt^2} + \frac{d^2\vec{x}'(t)}{dt^2} \tag{3}$$

$$\tag{4}$$

Therefore the accelerations of a body, $\vec{a}$ and $\vec{a}\,'(t)$, in the two frames are related by

$$\vec{a} = +\vec{f} + \vec{a}\,'(t) \tag{5}$$

where $\vec{f}$ denotes the acceleration of the frame K' w.r.t. the frame K. Multiplying the above equation by the mass of the body

$$M\vec{a} = M\vec{f} + M\vec{a}'(t) \tag{6}$$

$$\text{or} \qquad M\vec{a}' = M\vec{a}(t) - M\vec{f} \tag{7}$$

$$\tag{8}$$

Using Newton's Laws in the first frame K and replacing $M\vec{a}(t)$ with force $\vec{F}$ we get

$$\text{or} \qquad M\vec{a}' = \vec{F} - M\vec{f} \tag{9}$$

Pseudoforces

The second term $-M\vec{f}$ is called pseudo force. The EOM in the non inertial frame look similar but one has to include the pseudo force to the forces acting on the system. The pseudo forces to be included will depend on how the frame is accelerating. The pseudo force is equal to minus of mass times acceleration of the frame with respect to an inertial frame.

You may ask how are pseudo forces different from real forces. Can one distinguish between real and pseudo forces experimentally? Before answering this question I have to briefly concepts of gravitational mass and inertial mass.

It turns out that it is locally impossible to distinguish between gravitational forces and pseudo forces. This is because the both gravitational forces and pseudo forces in an inertial frame are proportional to the mass of the body. In practical terms it means that effect of gravitational forces in a given inertial frame can be reproduced by that coming from pseudo forces in an non inertial frame. So for example, EOM in uniform gravitational field is same as motion in a lift accelerating upwards without any gravitational forces. In his form it is known as Einstein's equivalence principle, and this statement holds for small regions of space.

It should be noted that the pseudo forces on a body are always proportional to the mass of a body.

§3 Examples of Accelerated Systems

If we have an accelerated frame of reference, like an accelerating elevator a car or train, the acceleration of the frame can be measured by doing a mechanical experiment entirely within the elevator or thrcar or the train without need for looking outside.This is possible because the equations of motion take different forms in differently accelerating frames. One might be tempted to ask w.r.t.the frame of reference w.r.t. the acceleration is being measured. The answer is very simple the acceleration is w.r.t. an inertial frame; and independent of the inertial we frame we may have in mind.

§4 Gravitational and Inertial Masses

A force $\vec{F}$ acting on a body produces an acceleration. By Newton's second law we have force proportional to acceleration of the body. Thus $\vec{F} = m_i\vec{a}$, where m_i is a property of the body; we call this inertial mass.

It is known that the gravitational pull $vecP$ of a body A (mass M) on another body is proportional to the mass m_g of the second body and we can write $P \propto m_g$. The body A will exert double gravitational pull on a body of mass $2m_g$, and so on.

Given the two situations above, it turns out that the m_i, m_g are some properties of the second body. There is no reason to expect a relation between the two inertial and gravitational masses of a body.

In order to understand the point being made above, note that the electrical force due to a charge Q on a second charge is proportional to the charge q of the second body. Here charge q plays the role played by m_g for gravitational force, and that the charge q is in no way related to the inertial mass.

It turns out that the ratio of inertial and gravitational masses is same for all bodies and is equal to one. The equality of gravitational and inertial masses has been experimentally verified to a great accuracy.

That these two masses are different can also be understood by asking how canwe measure inertial mass; how can we measure gravitational mass.

A simple arrangement for measurement of inertial masses is shown in figure below.

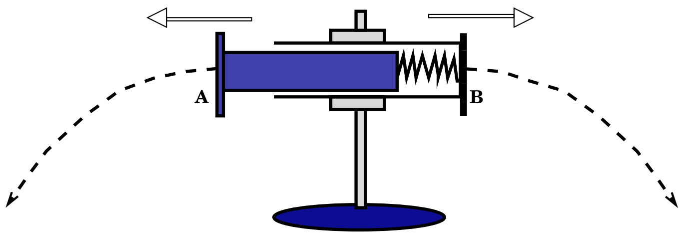

Fig. **1**

`Setup borrowed from IIT Kanpur B.Tech. Lab`

Here A, B are two cylinders mounted on a stand. A spring attached to the inside the hollow cylinder. A nail holds the two cylinders in place with spring in compressed position. When the nail is removed the two cylinders fly apart in opposite directions. The horizontal components of their momenta,p_A, P_B, will be equal and opposite at all times before hitting the ground, $P_A = P_B$. Their velocities are related by $M^i_B V_A = M^i_B V_B$. Here M^i_A, M^i_B denote the inertial masses of the two bodies. A video clip of the motion of the two cylinders can be made and analyzed with a suitable software and horizontal components of velocities can be measured. Taking mass of one of the bodies as unit, inertial mass of the other body is now determined.

One can use a spring balance to determine the gravitational pull on different bodies gives a scheme of measurement of gravitational masses.

It turns out that the ratio of gravitational and inertial masses is same for all bodies and therefore, with suitable choice of units, the ratio can be taken as unity. Several precise experiments have been done and the ratio of gravitational and inertial masses is known to to be unity with an error of about 10^{-12}.

§5 Pseudoforce vs Gravitational Force; Equivalence Principle

An important consequence of equality of gravitational and inertial masses is that it is impossible to distinguish by doing any mechanical experiment.

(a) The frame of reference is a non inertial frame moving with with some acceleration $\vec{a}$;

(b) The frame of reference is an inertial frame and there is an extra gravitational field $-m\vec{a}$ acting on bodies inside the elevator.

This is most easily seen by writing equation of motion in the two cases above. Let us consider ball suspended with a spring inside an elevator. The elevator in turn is accelerating upwards with constant acceleration a. Viewed by an observer outside the elevator and, positioned in a inertial frame, the acceleration of the ball is a upwards. The equation of motion of the ball using Newton's laws in the inertial frame takes the form.

$$\text{Mass} \times \text{Accn} = \text{Force} \Longrightarrow Ma = k\Delta\ell - Mg \tag{10}$$

The amount $\Delta\ell$ is the change in length when the ball hangs in equilibrium in the elevator's frame of reference.

As viewed from an inertial frame, it can be interpreted as if there is an extra downwards (pseudo) force equal to Ma which in turn can be seen to be equivalent to a downward gravitational pull of Ma.

As a result for an observer in the frame of the elevator, it becomes impossible to distinguish between the following two interpretations of the equation of motion.
One view is that the elevator is accelerated and the pseudo force have is to be included in the equation of motion.
The other view is that the elevator is not accelerated, but there is a gravitational pull $W' = ma$ in the direction opposite t acceleration in addition to the usual weight of the body $W = Mg$.

No mechanical experiment done entirely within the elevator can differentiate between the above two possibilities.

In other words effects of gravitational fields can be mimicked by going to an accelerated frame and having no gravitational field. This is possible only in small regions of space time. This is essence of the *equivalence principle*. In the form stated here it applies to mechanical systems.

A stronger version of equivalence principle states that all laws of Physics respect the equivalence principle. The equivalence principle played an important role in development of general relativity. [1]

§6 EndNotes

§6.1 Food For Your Thought

1. Air and He balloons in a cart.

   A cart carries an air ballon suspended from ceiling and a He filled balloon tied to the floor. The figure shows the situation when the cart is moving with uniform velocity. How will the two ballons appear when the cart has an non zero acceleration in the forward direction indicated in the figure by an arrow?

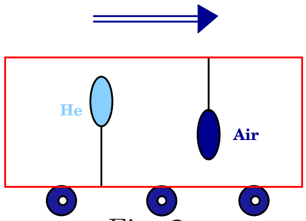

Fig. **2**

   Which of the following figures, (a),..,(d), depicts the correct situation when the cart accelerates as above.

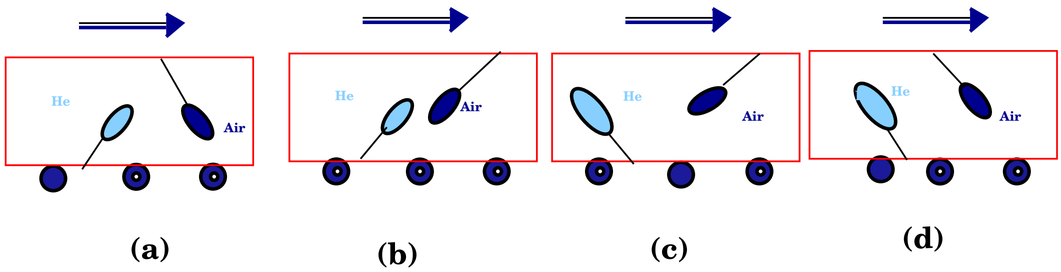

Fig. **3**

§6.2 Recommended for Further Reading

- Several experiments have been performed to verify equality of inertial and gravitational masses. Galileo's famous experiment (1589-92), of dropping different objects from leaning tower of Pisa, demonstrates just this.
  Learn more about Pisa tower experiment

- In 1889, the Hungarian physicist Roland Etövös used a slightly different approach to verify the equivalence of gravitational and inertial mass for various substances to an accuracy of about 10^{-8}, and the best such experiment improved on even this phenomenal accuracy, bringing it to the 10^{-12} level accuracy.
  Click to learn more about this equivalence

- This equality is formulated as equivalence principle which is an important principle for general relativity.

Click to know more about weak and strong forms of equivalence principle.
Click to learn more about equivalence principle.

- Perform a google search and learn more about it.
- Weinberg [1]discusses inertial and gravitational masses, Etövös experiment and role of equivalence principle in general relativity.
  Borrow the book from library or else click here

§6.3 Watch A Video

Equality of inertial and gravitational masses has been tested to great accuracy. Here is video demonstration of equality of inertial and gravitational masses. An experiment involving a feather and a ball was performed in a big hall A feather and a ball are dropped from same height. The ball reaches the floor first, the feather takes much longer. This difference in time of free fall is attributed to large air friction acting on the feather. The same experiment is repeated in the hall after evacuating the hall.

Brian Cox visits NASAs Space Power Facility in Ohio to see what happens when a bowling ball and a feather are dropped together under the conditions of outer space.

The result is for you to see.

Click here to watch the video on Youtube   Watch the video on BBC site

References

[1] Weinberg S. *Gravitation and Cosmology.* John Wiley & Sons,Inc. New York, First edition, 1972.