QFT-03 Tutorial and Assignments

Second Quantization of Schrodinger Equation

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§1 Assignment

Ladder Operators

About This Set

- You need to recall what you have already learned in harmonic oscillator.
- This set introduces different orderings (normal ordering, and more...) for operator products.
- This set requires you to apply basic commutation relations and properties of ladder operators to
- rearrange products of operators
- computation of matrix elements of between harmonic oscillator states.
- The techniques mastered here will be required in the tutorial on Rutherford scattering.

[1] Let $|ni\rangle$ denote the n^{th} excited state of a harmonic oscillator. Show that [2]

$$\langle n|x|m\rangle = \sqrt{\frac{\hbar}{2m\omega}}\Big(\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\Big) \tag{1}$$

[2] Using the properties of the ladder operators $a, a^\dagger$ and the number operator N, compute the average values of kinetic and potential energies for a harmonic oscillator in the n^{th} state $|n\rangle$. Verify that their sum equals $(n + 1/2)\hbar\omega$. [4]

[3] Rearrange the operator expression $pqpq^2$ as sum of expressions of the form $\sum_{m,n} c_{mn} q_n p_m$ in which each term has all q operators on the left and all p operators on the right.

[4] A normal product of operators $a, a^\dagger$ is an expression having all raising operators on the left and all lowering operators on the right. As an example, $aa^\dagger$ can be written as $a^\dagger a + 1$, where the operators appear in the normal product form. What is the expectation value of such a product in the state $|0\rangle$? Express the following operators as a sum of operators in the normal product form and a constant. [4]

(a) $aa^{\dagger 2}a^2a^{\dagger 2}$, (b) $a^{\dagger 2}aa^{\dagger 2}a$.

[5] Assuming anticommutation relations

$$[a_m, a_n]_+ = 0, \quad [a_m, a_n^\dagger]_+ = \delta_{mn}, [a_m^\dagger, a_n^\dagger]_+ = 0,$$

Prove that the number operators $N_n = a_n^\dagger a_n$ have eigenvalues 0 or 1 only.

§2 Tutorial

Rutherford Scattering

About This Tutorial Problem in this tutorial concerns computation of differential cross section for Coulomb scattering using second quantized Schrödinger field theory in the first order of perturbation theory in the interaction picture.
To help you formulate the problem, I will give you steps in this computation. You have to fill in details.

I divide the steps into three parts for purpose of this tutorial. My list of steps as given here is long and detailed. Once you have understood the computation formulate your own steps and rules.

Part-I::Calculating the matrix element

[1] The Lagrangian density for Schrodinger field is

$$\mathscr{L} = i\hbar\psi^*(\vec{r},t)\frac{\partial}{\partial t}\psi(\vec{r},t) + \frac{\hbar^2}{2m}\Big(\nabla\psi^*(\vec{r},t)\Big)\Big(\nabla\psi(\vec{r},t)\Big) - \psi^*(\vec{r},t)V(\vec{r})\psi(\vec{r},t) \qquad (2)$$

where $V(\vec{r})$ external potential potential seen by the Schrodinger field. You will get the same result that is obtained by using Fermi Golden rule and Born approximation in Schrodinger Wave mechanics.

At the end you compute the Rutherford scattering of alpha particles from a nucleus of charge Z.

[2] In the second quantisation scheme in interaction picture, expand the field in terms of plane waves and treat the expansion coefficients $a(\vec{k})$ as

$$\psi(\vec{r},t) = \int_\infty^\infty a(\vec{k})e^{i\vec{k}\cdot\vec{r}}\,d^3k \qquad (3)$$

$$\psi^*(\vec{r},t) = \int_\infty^\infty a^\dagger(\vec{k})e^{i\vec{k}\cdot\vec{r}}\,d^3k \qquad (4)$$

the ETCR for the operators $a(\vec{k})$ and $a^\dagger(\vec{k})$ are given

$$\big[a(k), a^\dagger(q)\big] = \delta(\vec{k}-\vec{q}) \qquad (5)$$

[3] In interaction picture

$$H'_I = \exp(iH_0t/\hbar)H'\exp(-iH_0t/\hbar)$$

where H_0 is free Hamiltonian and

$$H' = \int d^3x\psi^*(\vec{r},t)V(\vec{r})\psi(\vec{r},t)$$

Next compute the transition amplitude for the scattering by computing the matrix elements of the first order term in the perturbation series of the time evolution operator. Thus compute

$$m_{fi} = \int_{-T/2}^{T/2} \langle f|\, H_{\mathrm{I}}'(t)\, |i\rangle \, dt \tag{6}$$

Here $|i\rangle$ and $|f\rangle$ denote the initial and final states of a single particle, with momenta $\vec{k}_i, \vec{k}_f$.

Use the fact that initial and final states are eigenstates of free Hamiltonian with energies $E_i = \hbar^2 k_i^2/2m, E_f = \hbar^2 k_f^2/2m$ respectively. Integrate over time and obtain the transition amplitude $m_{\rm fi}$ This should give a factor

$$\frac{\sin(\Delta E)T/2)}{\Delta ET/2}, \qquad \text{where } \Delta E = E_{\rm i} - E_{\rm f}.$$

This factor has to handled carefully as done in the class *after squaring the amplitude.*

[4] Use

$$|i\rangle = a^\dagger(\vec{k}_i)|0\rangle, \qquad |f\rangle = a^\dagger(\vec{k}_f)|0\rangle,$$

and obtain the matrix element as function of momenta. Keep the integral over $V(r)$ as it is, don't try to do the space integral at this stage.

Part-II ::Transition probability per unit time

Next square and compute the transition probability. It will have square of the factor $\frac{\sin(\Delta E)T/2)}{\Delta ET/2}$ as given above.

Find transition probability per unit time, differentiate $|m_{\rm fi}|^2$ w.r.t. time T and take limit $\Delta T \to \infty$. In this limit you should get a Dirac delta function $\delta(E_i - E_f)$.

Denote this transition probability per unit time as $w_{\rm fi}$

Part-III :: Find Cross sections and Life Times

[1] In a measurement of differential cross section all particles with for final momenta in the range $\vec{k}, \vec{k} + d\vec{k}$ are counted. Therefore we must integrate over all momenta in this range.

Change notation $k_f \to k$,

Relate the range d^3k to the solid angle as $k^2 dk\, d\omega$.
Use Dirac delta function to carry out the integral over k.

Remember delta function makes energies equal $E_f = E_i$, therefore $\therefore k_i = k_f = k$.

[2] Imagine the scattering experiment is repeated by sending one particle N times, then
`Number of particles scattered in to solid angle` $d\Omega$ `per sec`
`= N` $\times$ `transition probability per unit time to states`
`with final momentum in the solid angle` $d\Omega$.

Use the fact, that for differential cross section, the range of solid angle is small. So that integration over $d\Omega$ results in multiplying by $d\Omega$.

[3] `The number of particles scattered into solid angle` $d\Omega$
$=$ `Flux` $\times\, d\Omega\times$ differential cross section,
and

`Flux =` $N\times$ `probability current`
where N is the total number of particles sent during the experiment. .

[4] Use (7) and (8) to find the differential cross section. Replace $V(r) = Ze^2 \exp(-\mu r)/r$ and compute the integral over spatial coordinates now.

[5] Take the limit $\mu \to 0$ and compare your answer with Rutherford formula for Coulomb scattering.

This division into three major steps will remain the same for all situations that we are going to deal with. Details will differ from case to case.

§3 Assignment

Using Equations of Motion

About this assignment This assignment concerns the following

- Solving Heisenberg equations of motion
- Calculating unequal time commutator
- Computing vacuum expectation value of time ordered product

[1] (a) For harmonic oscillator

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

set up and solve up the equations of motion for position and momentum operators in the Heisenberg picture. Hence show that

$$\hat{x}(t) = \tilde{x}\cos\omega t + \frac{1}{m\omega}\tilde{p}\sin\omega t \tag{7}$$

$$\hat{p}(t) = \tilde{p}\cos\omega t - m\omega\tilde{x}\sin\omega t \tag{8}$$

where $\tilde{x}, \tilde{p}$ in the right hand sides of the above equations are the Schrödinger picture operators.

(b) Calculate the commutators

$$[\hat{x}(t), \hat{x}(t'))], \qquad [\hat{x}(t), \hat{p}(t'))], \qquad [\hat{p}(t), \hat{p}(t'))],$$

for $t \neq t'$. How do these commutators at unequal times compare with equal time commutators? Are $\hat{x}(t), \hat{x}(t')$ compatible observables?

(c) Verify that at equal times the commutators reduce to canonical commutation rule $[\hat{x}, \hat{p}] = i\hbar$.

[2] You have already obtained the expressions for position and momentum operators for arbitrary times. Compute the vacuumexpectaion value of the time ordered product $\langle 0|T(q(t_1)q(t_2))|0\rangle$.

For this problem, explain the steps that you will follow. Write different expressions that you will need to compute but you need not complete all mathematical details at this stage.

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