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Lectures on Scattering Theory

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§1 Electron scattering from atoms

An excellent example where Born approximation works is the elastic scattering of high energy electrons from neutral atoms. At these energies the exchange effects arising from indistinguishability of the incident electrons from the atomic electrons are negligible. The energies of interest are in the range of a few hundred electron volts.

The charge distribution of an atom consists of two parts: the positive point nucleus of charge Ze and the negative charge density ρ due to the atomic electrons.[1]

$$\rho_{\text{atom}} = eZ\delta^{(3)}(\vec{r}) - e\rho(r), \tag{1}$$

where, because of neutrality of the atom

$$\iiint_{-\infty}^{\infty} \rho(r) d^3r = Z. \tag{2}$$

We shall introduce the atomic form factor which is the Fourier transform of the electron density

$$F(q) \stackrel{\text{def}}{=} \iiint_{-\infty}^{\infty} d^3r e^{-i\vec{q}\cdot\vec{r}} \rho(r). \tag{3}$$

$$F(0) = Z. \tag{4}$$

[1] For simplicity we assume ρ to be spherically symmetric.

The potential $V(r)$ as seen by the electrons is

$$V(r) = -e\phi(r) \tag{5}$$

where $\phi(r)$ is the electrostatic potential corresponding to the charge density of the entire atom and is given by the Poisson's equation

$$\begin{aligned} \nabla^2\phi &= -4\pi\rho_{\text{atom}} && (6) \\ \nabla^2 V &= +4\pi e^2[Z\delta^{(3)}(\vec{r}) - \rho(r)]). && (7) \end{aligned}$$

The scattering amplitude in the Born approximation consists in taking the Fourier transform of the potential with respect to the momentum transfer $\vec{q}$. For this consider the Fourier transform of Eq.(7) straightaway by multiplying it by $e^{-i\vec{q}\cdot\vec{r}}$ and integrating over $\vec{r}$:

$$\begin{aligned} \iiint_{-\infty}^{\infty} d^3re^{-i\vec{q}\cdot\vec{r}}\nabla^2 V &= +4\pi e^2[Z - \iiint_{-\infty}^{\infty} d^3re^{-i\vec{q}\cdot\vec{r}}\rho(r)]. && (8) \\ -q^2\iiint_{-\infty}^{\infty} \nabla^2 V &= +4\pi e^2[Z - F(q)], && (9) \\ \iiint_{-\infty}^{\infty} \nabla^2 V &= -\frac{4\pi e^2}{q^2}[Z - F(q)], && (10) \end{aligned}$$

The scattering amplitude in Born approximation is

$$\begin{aligned} f(\theta) &= -\frac{m}{2\pi}\int e^{-i\vec{q}\cdot\vec{r}}V(r)d^3r && (11) \\ &= \frac{2me^2}{q^2}[Z - F(q)]. && (12) \end{aligned}$$

Hence the differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4m^2e^4}{q^4}\big(Z - F(q)\big)^2. \tag{13}$$

where $q^2 = 4k^2\sin^2\frac{\theta}{2}$.

From the observed values of $\frac{d\sigma}{d\Omega}$ we can determine the mean square radius,$\langle r^2\rangle$ of atomic charge distribution. defined by

$$\langle r^2\rangle \stackrel{\text{def}}{=} \frac{1}{Z}\iiint_{-\infty}^{\infty} r^2\rho(r)d^3r. \tag{14}$$

In terms of the form factor $F(q)$ we have

$$F(q) = Z(1 - \frac{\langle r^2\rangle}{6}q^2 + \cdots). \tag{15}$$

The charge density due to electrons in the atom will be taken as decreasing exponential (recall for the H atom $\psi^*\psi \sim e^{-r/a}$)

$$\rho(r) = \Big(\frac{Z}{8\pi R^3}\Big) e^{-\frac{r}{R}} \tag{16}$$

which implies

$$F(q) = \frac{Z}{(1+R^2q^2)^2}. \tag{17}$$

Incidentally, this type of form factor is called 'dipole form factor' in particle physics jargon. The electromagnetic form factor of proton as deduced from the experiments of elastic scattering of electrons on protons is simulated quite well by such a functional form.)

We finally have

$$\frac{d\sigma}{d\Omega} = \frac{4m^2Z^2e^4}{q^4}\Big[1 - \frac{1}{(1+R^2q^2)^2}\Big]^2. \tag{18}$$

Large scattering angle: Recall that we are discussing electron scattering at high energies where Born approximation is valid,

$$R^2q^2 = 4k^2R^2\sin^2\frac{\theta}{2} >> 1, \tag{19}$$

so that $F(q) << Z$; the effect of atomic electrons is negligible for scattering at angles and the effect due to the point nucleus alone is important. Thus we get the Rutherford celebrated scattering formula

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{Z^2m^2e^4}{4k^4\sin^4\frac{\theta}{2}}.} \tag{20}$$

In the text books we meet this formula in context of Coulomb scattering by a point charge Ze in Born approximation in the limit of screening radius $R \to \infty$. Here we derived the same formula by fixing R, (target atom fixes R) but confining our attention to scattering at large angles. Both limits clearly give the same answer because both satisfy $q^2R^2 >> 1$.

It is remarkable fact that the Rutherford formula which we obtained in Born approximation is identical to the cross-section formula obtainable in classical mechanics (as Rutherford originally did, see Goldstein's [**?**] book for this) as well as by exact non-relativistic treatment (*i.e.* without Born approximation) of the Coulomb potential. R. G. Newton remarks,"We can only speculate for how long the discovery of atomic nucleus by Rutherford would have been delayed if it had not been for this coincidence which made the classical argument used by Rutherford correct [**?**].

General comment: If the point nucleus were to be replaced by a slightly *extended charge distribution* then the above formula will be altered depending on whether the incident charge repels of attracts the nuclear charge. For the case of repulsion (as in Rutherford experiment) there is a distance of closest approach given by $r_{\min} = \frac{Ze^2}{E}$ and hence, if the radius of the nuclear charge distribution is smaller than $r_{\min}$ then the cross section is unaltered. For the case of attraction (as with electron beams on nuclei, Hoafstadter exepriment), the distance of closest approach is zero and all the incident particles go right through the center; hence they *always* explore the charge distribution of the nucleus.

Small scattering angles we have $R^2q^2 << 1$ and

$$F(q) \approx Z(1 - 2R^2q^2) \tag{21}$$

so that Eq.(18) becomes

$$\frac{d\sigma}{d\Omega} \cong 16Z^2m^2e^4R^4 \tag{22}$$

which is independent of momentum and angle. In particular at $\underline{\theta = 0}$ there is no divergence. On the other hand it is well-known that the scattering cross-section of a charged particle by a point nucleus according to the Rutherford formula, (20), diverges at $\theta = 0$; whereas according to Eq.(22) this divergence is absent because of the *effect of screening* by atomic electrons which enters through the function $F(q)$. This agrees with our intuition that small deflections of the incident particles correspond to large impact parameters.

Comparison with experiments

We quote only some recent results. Experiments of electrons on Helium a 500 eV are in excellent agreement with Born approximation except at small angles (see J.P. Bromberg, J.Chem. Phys. **50** (1969) 3906)). The cause of disagreement at small angles is attributed to the neglect of distortion of the atom due to the incident electrons and exchange effects.

In this connection it is worth mentioning that electron scattering on the simplest of atoms, namely H atom, has not been carried out until a few years ago. This because the naturally occurring hydrogen is molecular, and one needs atomic beams of high intensity to do the experiments. Recently such experiments have indeed been carried out (C.R. Lloyd et al., Phys. Rev. **A10** (1974) 175) for $eH \to eH$ at electron energies of 100 eV, 200 eV, etc. The verdict is Born approximation seems to be too low by about 20-40% at 200 eV for $15^o < \theta < 135^o$.