QM-11 Lecture Notes
Time Dependent Schrödinger Equation
Solution for Wave function at time t *

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We assume that the Hamiltonian of the particle is independent of time and that it can be written in the form

$$H = \frac{p^2}{2m} + V(\vec{r}) \tag{1}$$

The Schrodinger equation for the particle moving in potential $V(\vec{r})$ can be written as

$$-i\hbar\frac{\partial\psi(t)}{\partial t} = \frac{1}{2M}\nabla^2\psi(t) + V(\vec{r})\psi(t). \tag{2}$$

The possible states of the particle at a time t will be represented by square integrable wave function wave function $\psi(\vec{r}, t)$.

When time does not appear in the hamiltonian of a system, the equation of motion can be solved by the method of separation of variables. Thus by substituting

$$\psi(\vec{r}, t) = u(\vec{r})T(t) \tag{3}$$

in Eq.(**??**) we get

$$i\hbar\frac{1}{T(t)}\frac{dT(t)}{dt} = \hat{H}u(\vec{r}) \tag{4}$$

Equating each side to a constant, say E we get two equations for $u(\vec{r})$, and for $T(t)$, as follows.

$$i\hbar\frac{d}{dt}T(t) = ET(t) \tag{5}$$

$$\hat{H}u(\vec{r}) = Eu(\vec{r}) \tag{6}$$

Let $u_1(\vec{r}), u_2(\vec{r}), ...$ denote the eigenvectors of the Hamiltonian $\hat{H}$ with the eigenvalues $E_1, E_2, ...$, respectively.

$$\hat{H}u_k(\vec{r}) = E_k u_k(\vec{r}); \qquad k = 1, 2, 3, ... \tag{7}$$

* ver 1.x; DateCreated: 2015

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Eq.(**??**) and Eq.(**??**) have solutions given by

$$T_k(t) = \text{const} \times \exp(-iE_k t/\hbar), \qquad Hu_k(\vec{r}) = E_k u_k(\vec{r}) \tag{8}$$

and (**??**) has *an infinite number of solutions*, one for each real k, given by

$$\phi_k(\vec{r}, t) = u_k(\vec{r}, t) \exp(-iE_k t/\hbar) \tag{9}$$

and the most general solution is a linear combination of solutions $\phi_k(\vec{r}, t)$ in Eq.(**??**) and is given by

$$\psi(\vec{r}, t) = \sum_{k=1}^{\infty} c_k u_k(\vec{r}) \exp(-iE_k t)/\hbar). \tag{10}$$

If the wave function at time t_0, is $\psi(\vec{r}, t_0) \equiv \psi_o(\vec{r})$, the expression (**??**) evaluated at $t = t_0$ gives

$$\psi_0(\vec{r}) = \sum_{k=1}^{\infty} c_k u_k(\vec{r}) \exp(-iE_k t_0)/\hbar) \tag{11}$$

Using the orthogonality of energy eigenfunctions we can find the coefficients α_k and are given by.

$$(u_k, \psi_0) = c_k \exp(-iE_k t_0)/\hbar) \tag{12}$$

and the knowledge of the coefficients c_k allows us to compute the wave function at time t from (**??**). The final answer for the wave function at time t is given by

$$\psi(\vec{r}, t) = \sum_{k=1}^{\infty} c_k u_k(\vec{r}) \exp(-iE_k t)/\hbar)\psi(\vec{r}, t) = \sum_{k=1}^{\infty} (u_k, \psi_0) \exp(-iE_k(t-t_0)/\hbar)\, u_k(\vec{r}) \tag{13}$$

ψ_0 is the wave function of the system at time t_0.

$\boxed{\text{Flagged for adding a section on stationary states}}$