Lecture 2
Maxwell's Equations

Was it a God who wrote these lines ...?

Ludwig Boltzmann

(*Vorlesungen uber Maxwells Theorie der Electrizitat und des Lichtes*,Vol II, Munchen, 1893)

2.1 The Basic Fields E and B

Electric and magnetic fields are two facets of the same field called electromagnetic field (em-field for short). The em-field is created by charge and current densities (ρ and $\mathbf{j}$ respectively). They are governed by **Maxwell's equations** :

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \text{(Gauss' Law)} \tag{22}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{23}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(Faraday's Law)} \tag{24}$$

$$c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t}. \tag{25}$$

These equations are supposed to determine $\mathbf{E}$ and $\mathbf{B}$ which are six quantities dependent on space-time. These are eight equations (two scalar and two vector equations). Are these over-determined?

2.2 Conservation of charge

Surely, the Maxwell's equations do not hold for any arbitrary independent values of ρ and $\mathbf{j}$. Taking the dot product with ∇ in the last of the Maxwell equations the left hand side is zero because $\nabla.(\nabla \times (\text{any vector})) = 0$ and the right hand side is

$$\frac{1}{\epsilon_0}\nabla \cdot \mathbf{j} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E})$$

If we substitute from the first of the equations we get the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{26}$$

which simply tells us that in a volume V with its surface S the depletion of charge per unit tine is accounted for by flow of charge from the surface :

$$\int_S \mathbf{j}.\hat{\mathbf{n}} da = \int_V \nabla \cdot \mathbf{j} dv = -\frac{\partial}{\partial t}\int_V \rho dv$$

2.3 Poincare Lemmas

We know that $\mathbf{A}$ is any vector field then

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0.$$

What is interesting is that, the converse is also true, at least in a 'small region'. If we find that for some vector field $\mathbf{B}$

$$\nabla.\mathbf{B} = 0$$

then $\mathbf{B}$ must be of the form $\nabla \times \mathbf{A}$ in a neighbourhood.

Similarly, we know that $\nabla \times (\nabla\psi)$ is always zero. The Poincare lemma for this case says that if for some vector field $\mathbf{C}$ it is true that $\nabla \times \mathbf{C} = 0$ then in a small region there exists a scalar function ϕ such that we can express $\mathbf{C} = \nabla\phi$.

These lemmas are very useful. The 'small region' or neighbourhood is very often the whole space in simple cases which we deal with. The exceptions will be discussed separately.

2.4 Potentials

Two of the Maxwell equations do not involve any charge or current. They are

$$\nabla \cdot \mathbf{B} = 0, \quad \text{and} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Using the Poincare lemmas we can first write

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

This vector field is called **vector potential** or sometimes *magnetic* vector potential. Once $\mathbf{B}$ is so determined, we can put it in the Faraday's law

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{27}$$

where the Poincare lemma can be used again and we write

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi.$$

The field ϕ is called the **scalar potential**. The negative sign is traditional and has the same origin as the mechanical equation 'force = - gradient of potential'.

2.5 Gauge freedom

Electromagnetic fields do not determine the potentials completely. If magnetic induction **B** is given to us then vector potential **A** and $\mathbf{A}' \equiv \mathbf{A} + \nabla\Psi$ give the same field **B** because $\nabla \times \nabla\Psi = 0$. The electric field is

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi - \frac{\partial \mathbf{A}'}{\partial t} + \frac{\partial}{\partial t}\nabla\Psi$$

therefore if we choose a new scalar potential $\phi' = \phi - \partial\Psi/\partial t$ then even the electric field does not change :

$$\mathbf{E} = -\nabla\phi' - \frac{\partial \mathbf{A}'}{\partial t}$$

The transformation

$$\mathbf{A} \to \mathbf{A} + \nabla\Psi, \qquad \phi \to \phi - \partial\Psi/\partial t$$

is called a **gauge transformation** and the scalar function Ψ which governs the gauge transformation as the **gauge function**.

2.6 Gauge Fixing

It is simpler to work with potentials rather than em-fields **E** and **B** because in place of the two vector fields we can deal with only one vector field **A** and one scalar field ϕ. Moreover, we can forget about two of the Maxwell equations because they are automatically satisfied.

In terms of the potentials, the remaining Maxwell equations look like

$$\begin{aligned} \nabla^2\phi + \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} &= -\frac{\rho}{\epsilon_0} \\ c^2\nabla(\nabla \cdot \mathbf{A}) - c^2\nabla^2\mathbf{A} &= \frac{\mathbf{j}}{\epsilon_0} - \nabla\left(\frac{\partial\phi}{\partial t}\right) - \frac{\partial^2\mathbf{A}}{\partial t^2} \end{aligned}$$

which can be rearranged as

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} - \frac{\partial}{\partial t}\nabla\cdot\mathbf{A}, \quad (28)$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\frac{\mathbf{j}}{\epsilon_0 c^2} + \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right). \quad (29)$$

The non-uniqueness of potentials is not a big problem. We can restrict the freedom in the choice of potentials by imposing restrictions of our own on the potentials. Such restrictions may 'fix' the potentials wholly or partly. We can also choose the condition on potentials conveniently to simplify our equations.

Out of the infinitely many ways of gauge fixing there are two popular choices.

1. We impose

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0. \quad (30)$$

This choice is called the **Lorentz gauge** and the Maxwell equations become

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -\frac{\rho}{\epsilon_0}, \quad (31)$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\frac{\mathbf{j}}{\epsilon_0 c^2}. \quad (32)$$

2. We require

$$\nabla\cdot\mathbf{A} = 0, \quad (33)$$

and this choice is called the **Coulomb gauge**. In this case the Maxwell equations look like

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \quad (34)$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\frac{\mathbf{j}}{\epsilon_0 c^2} + \frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t} \quad (35)$$