QM-20 Problem Solving Units
Spin and Identical Particles

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UNIT-I

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§1 Introduction

Spin of a particle can be defined as a new dynamical variables which is like angular momentum and commutes with position and momentum of the particles. In this unit you will learn about writing, interpreting and using spin wave function of a particle in variety of situations.

The spin of a particle, like any other dynamical variable, can be represented by hermitian matrices. You will learn how to construct spin matrices, their properties and their uses.

§2 Problem Solving Objectives

After reading this unit you should be able to

1. To construct spin operators for any spin. To learn some common properties of spin operators.

2. To use information about spin state of a particle to obtain its spin wave function.

3. To use the wave function to derive specific information about the state of the system.

4. To obtain time development of spin systems.

§3 Road map

It is important to revise the postulates of quantum mechanics and recall the properties of Pauli matrices.These are reproduced in the two appendices. Revise and get started.

1. The properties of spin operators for a given spin, s, and their matrix representation are obtained by using the commutation properties. The spin operators have the same commutation relations as angular momentum operators.*It will be very helpful if you revise angular momentum.* The most useful properties of spin operators are listed below.

   (a) The operator $\vec{S}^2$ commutes with each component of spin S_x, S_y and S_z. The operators $\vec{S}^2$ and any one component of spin have complete set of simultaneous eigenvectors $|sm\rangle$:

$$\vec{S}^2|sm\rangle = s(s+1)\hbar^2|sm\rangle, \qquad S_z|sm\rangle = m\hbar|sm\rangle \tag{1}$$

(b) For a particle of spin s the eigenvalues of S^2 are $s(s+1)\hbar^2$ and eigenvalues of each of S_x, S_y, S_z are $s\hbar, (s-1)\hbar, (s-2)\hbar, \ldots - s\hbar$. The above statement is also true of $\hat{n} \cdot \vec{S}$ for a unit vector $\hat{n}$. (See item # 3 below)

(c) The operators $S_\pm \equiv S_x \pm iS_y$ act like raising and lowering operators for S_z values.

$$S_\pm|m\rangle = \sqrt{s(s+1) - m(m \pm 1)}\hbar|m \pm 1\rangle$$

(d) The action of S_+ on the highest spin projection state $|m = s\rangle$and S_- on the lowest spin projection state$|m = -s\rangle$, both, give zero.

2. The spin operators for a spin 1/2 particle are the Pauli matrices.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. The eigenvalues of S^2 are $s(s+1)\hbar^2$. If $\hat{n}$ is a unit vector, for a fixed value s, the operator $\hat{n} \cdot \hat{S}$ has $(2s+1)$ eigenvalues

$$s\hbar, (s-1)\hbar, \ldots (-s+1)\hbar, -s\hbar$$

from $s\hbar$ to $-s\hbar$ decreasing in steps of 1.
The set of eigenvalues of $\vec{\alpha} \cdot \vec{S}$ are obtained from **the same set {** $s\hbar, (s-1)\hbar, (s-2)\hbar, \ldots - s\hbar$**} by multiplying by** $|\alpha| \equiv \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$**.**

For solving problems in this unit an understanding of postulates of quantum mechanics is crucial. The relevant parts are reproduced at the end.

§4 Examples

Problem 1: Construct the spin matrices for a spin half particle in the basis in which S_z is diagonal.

Here it is useful to recall from vector spaces that in an orthonormal basis $\{|k\rangle, k = 1, 2, .., n\}$ *the matrix for an operator* $\hat{T}$ *in Dirac notation is given by*

$$\boxed{\underline{\mathsf{T}} = \begin{pmatrix} \langle 1|\hat{T}|1\rangle & \langle 1|\hat{T}|2\rangle & \ldots & \langle 1|\hat{T}|n\rangle \\ \langle 2|\hat{T}|1\rangle & \langle 2|\hat{T}|2\rangle & \ldots & \langle 2|\hat{T}|n\rangle \\ \ldots & \ldots & \ldots & \ldots \\ \langle n|\hat{T}|1\rangle & \langle n|\hat{T}|2\rangle & \ldots & \langle n|\hat{T}|n\rangle \end{pmatrix}} \tag{2}$$

☺*Solution*: For a spin $\frac{1}{2}$ particle, the S_z eigenvalues are $\frac{\hbar}{2}, -\frac{\hbar}{2}$. In the basis chosen $S-z$ is diagonal therefore it is straightforward to write the matrix for S_z

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

Next we will need to recall

$$S_\pm|m\rangle = \sqrt{s(s+1)-m(m\pm 1)}\hbar|m\pm 1\rangle$$

and use it for $s=\frac{1}{2}, m=\pm\frac{1}{2}$ to get

$$\begin{aligned}
S_+|\tfrac{1}{2}\rangle = 0 \quad &\Rightarrow \quad \langle\tfrac{1}{2}|S_+|\tfrac{1}{2}\rangle = 0, \qquad \langle-\tfrac{1}{2}|S_+|\tfrac{1}{2}\rangle = 0 && (4)\\
S_+|-\tfrac{1}{2}\rangle = |\tfrac{1}{2}\rangle \quad &\Rightarrow \quad \langle\tfrac{1}{2}|S_+|-\tfrac{1}{2}\rangle = 1, \qquad \langle-\tfrac{1}{2}|S_+|-\tfrac{1}{2}\rangle = 0 && (5)\\
& && (6)
\end{aligned}$$

Therefore, the matrix

$$\begin{pmatrix} \langle\frac{1}{2}|S_+|\frac{1}{2}\rangle & \langle-\frac{1}{2}|S_+|\frac{1}{2}\rangle \\ \langle\frac{1}{2}|S_+|-\frac{1}{2}\rangle & \langle\frac{1}{2}|S_+|\frac{1}{2}\rangle \end{pmatrix} \tag{7}$$

for S_+ becomes

$$S_+ = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{8}$$

The matrix for S_- is hermitian adjoint of S_+ and hence we get

$$S_- = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{9}$$

The matrices for, S_x, S_y, are therefore given by

$$S_x = \tfrac{1}{2}\Big(S_+ + S_-\Big) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{10}$$

$$S_y = \tfrac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{11}$$

To summarize we get the result that the spin $\frac{1}{2}$ matrices are given by

Problem 2: Find the spin wave function for a spin $\frac{1}{2}$ particle. It is given that spin projection along the x- axis has a definite value

(a)$\dfrac{\hbar}{2}$ (b) $-\dfrac{\hbar}{2}$

☺*Solution*: Since spin along the x axis has a definite value, the corresponding spin wave function will be eigenfunction of $S_x \equiv \frac{\hbar}{2}\sigma_x$.

(a) Thus the required wave function for $\hbar/2$ will be given by

$$S_x\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Longrightarrow \sigma_x\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{12}$$

$$\therefore \qquad a = b \tag{13}$$

Using this we get normalized eigenfunction as $\chi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) Similarly, for spin projection S_x equal to $-\hbar/2$ we would get

$$S_x\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Longrightarrow \sigma_x\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \tag{14}$$

$$\therefore \qquad a = -b \tag{15}$$

Using this we get normalized eigenfunction as $\chi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Problem 3: Spin wave function of a spin $\frac{1}{2}$ particle is given to be

$$|\phi\rangle = \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$$

(a) Find the ratio of probabilities that S_z has values $\pm\hbar/2$.

(b) Find the ratio of probabilities that S_x has values $\pm\hbar/2$.

(c) Compute the average value of spin projection along the direction (1,1,1).

☺*Solution*:

`Recall` If the system is in a state $|\Psi\rangle$, the probability of finding a value α for a dynamical variable A is given by $|\langle\alpha|\Psi\rangle|^2$.

(a) The eigenvalues and eigenvectors of S_z are

$$\frac{\hbar}{2}, \quad |\chi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad -\frac{\hbar}{2}, \quad |\chi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{16}$$

Hence the corresponding probabilities are

$$P(+1/2) = |\langle\chi_1|\phi\rangle|^2 = \left|\frac{3-4}{5\sqrt{2}}\right|^2 = \frac{9}{25} \tag{17}$$

$$P(-1/2) = |\langle\chi_2|\phi\rangle|^2 = \left|\frac{3+4}{5\sqrt{2}}\right|^2 = \frac{16}{25} \tag{18}$$

(b) The normalized eigenvectors of S_x are given by

$$\frac{\hbar}{2}, \quad |\chi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}; \qquad -\frac{\hbar}{2}, \quad |\chi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{19}$$

The corresponding probabilities are $P(\pm 1/2)$ are given by

$$P(+1/2) = |\langle\chi_1|\phi\rangle|^2 = \left|\frac{3-4}{5\sqrt{2}}\right|^2 = \frac{1}{50} \tag{20}$$

$$P(-1/2) = |\langle\chi_2|\phi\rangle|^2 = \left|\frac{3+4}{5\sqrt{2}}\right|^2 = \frac{49}{50} \tag{21}$$

(c) The unit vector along the given direction is $\frac{1}{\sqrt{3}}(1,1,1)$ and the projection of spin is

$$S_{111} = \frac{1}{\sqrt{3}}\frac{\hbar}{2}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$$

Therefore the average is given by

$$\langle S_{111}\rangle = \langle\phi|S_{111}|\phi\rangle \tag{22}$$

$$= \frac{1}{25\sqrt{3}}\frac{\hbar}{2}\begin{pmatrix} 3 & -4 \end{pmatrix}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{23}$$

$$= -\frac{31\hbar}{50\sqrt{3}} \tag{24}$$

Problem 4: The interactiion Hamiltonian of a spin half particle, placed in magnetic field along x- axis. is given to be $H = \gamma S_x$. If the spin of the particle points in the positive z direction, what is the probability that the spin would have flipped after time $t = \pi/2\gamma$?

☺*Solution*: The initial state is given by $|\uparrow\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and at time t the state will be

$$\chi(t) = \exp(-iHt/\hbar)\chi_0.$$

Now

$$e^{-iHt/\hbar} = e^{-i\gamma S_x t/\hbar} = \cos\alpha t\hat{I} - i\sigma_x \sin\alpha \tag{25}$$

$$= \begin{pmatrix} \cos\alpha & -i\sin\alpha \\ -i\sin\alpha & \cos\alpha \end{pmatrix} \tag{26}$$

where the notation $\alpha = \gamma t/2$ and the identity

$$e^{\theta\hat{n}\cdot\vec{\sigma}} = \cos\theta\hat{I} + i\hat{n}\cdot\vec{\sigma}\sin\theta, \qquad \hat{n} = \text{unit vector}$$

have been used. The required probability is absolute square of the amplitude

$$\langle\downarrow|\cos\alpha\hat{I} + i\sin\alpha\sigma_x|\uparrow\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha & -i\sin\alpha \\ -i\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{27}$$

$$= -i\sin\alpha \tag{28}$$

Therefore the probability of spin flip at time t is $\sin^2(\gamma t/2) = 1/4$.

§5 Exercise

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Spin Wave Function

[1] Find the spin wave function of a spin 1/2 particle with spin projection $\hbar/2$ along the vector $(3, 4, 0)$. Compute the probabilities of S_x, S_y and S_z having value $\hbar/2$.

[2] Spin wave function of a spin 1/2 particle is given to be

$$\begin{pmatrix} 12 \\ 7 + i10 \end{pmatrix}$$

(a) Find the ratio of probabilities that S_z has values $\pm\hbar/2$.

(b) Find the ratio of probabilities that S_x has values $\pm\hbar/2$.

(c) Write the operator for the spin projection along a vector (a, b, c)? and compute the average value of the spin projection along (1,1,1).

(d) For the general case of $S_n \equiv \hat{n} \cdot \vec{S}$, where $\hat{n}$ is a unit vector, prove that the (non-normalized) eigenvectors of S_n can be written as

$$\frac{\hbar}{2}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + n_3 \\ n_1 + in_2 \end{pmatrix}; \qquad -\frac{\hbar}{2}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} n_1 - in_2 \\ -1 - n_3 \end{pmatrix}. \tag{29}$$

[3] (a) For a spin $\frac{1}{2}$ particle find the state vector when the physical state is given to be the one having a sharp value of S_y equal to $-\hbar/2$.

(b) If the Hamiltonian of the system is $H = \gamma S_z$, find wave function of the system at time t if the initial spin state at time $t = 0$ is that given the first part of this question.

[4] The spin wave function of a spin $\frac{1}{2}$ particle is given to be

$$\chi = \begin{pmatrix} e^{i\alpha} \cos\delta \\ e^{i\beta} \sin\delta \end{pmatrix}.$$

Determine the polar angles θ, ϕ corresponding to the direction $\hat{n}$ along which the spin has a value $\frac{\hbar}{2}$. How do you interpret the result that the polar angles depend only on the difference $\alpha - \beta$ and not on α and β separately?

§6 Exercise

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Spin Matrices

[1] In the basis in which S_z is taken to be diagonal, show that the spin matrices for a spin one particle are given by

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hint : A useful result

$$S_\pm |m\rangle = \sqrt{s(s+1) - m(m \pm 1)}\hbar |m \pm 1\rangle$$

[2] Use results of the previous problem to answer the following questions for a spin one particle.

(a) What are the spin wave functions for the states in which S_z will have values $\hbar, 0, -\hbar$?

(b) If the spin wave function is given by

$$\chi = \begin{pmatrix} 12 \\ 4i \\ 8 - i6 \end{pmatrix}$$

Find the probabilities that the x component of spin will have values
(i) $\hbar$; (ii) 0; (iii) $-\hbar$.

[3] (a) Construct the spin matrices for spin $\frac{3}{2}$ particles.

(b) Find the matrix for $\vec{S}^2$ and verify that it is a multiple of identity matrix.

(c) What eigenvalues do you expect for S_x? Check your answer by an explicit computation of the eigenvalues using the matrices you have found.

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[1] What is the range of the eignevalues of $S_x + S_y$ for a spin 4 nucleus?

(a) only 4 (b) only -4 (c) -4 to + 4 (d) $-4\sqrt{2}$ to $4\sqrt{2}$

[2] What is the number of components of spin wave function of a spin 3/2 particle?

(a) 2 (b) 3 (c) 4 (d) 6

[3] In an answer to a question a student calculated average value of S_x for an electron and found a value $\frac{31\hbar}{25\sqrt{3}}$. Then

(a) the answer may or may not be correct; it depends on the spin wave function.

(b) the answer is certainly wrong. In this case give a short explanation.

[4] The spin half matrices are known to obey relations (in units $\hbar = 1$)

$$\vec{S}^2 = \frac{3}{4} I_2, \qquad 4X^2 - I_2 = 0,$$

where $S^2 = S_x + S_y + S_z, X = \hat{n}\vec{S}, \hat{n}$ is a unit vector, and I_2 is 2×2 identity matrix. Which of the following option gives the correct corresponding relations for spin 1 matrices

(a) $S^2 = I_3, \qquad X^2 - I_3 = 0;$ (b) $S^2 = \frac{3}{4} I_3, \qquad X^2 - X = 0;$

(c) $S^2 = 2I_3, \qquad X^3 - X = 0;$ (d) $S^2 = 2I_3, \qquad X^2 - X = 0$

where I_3 is 3×3 identity matrix.

§8 Questions

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Spin Wave Functions

[1] For a spin half particle the spin wave function has two components. What is the number of components of a spin $\frac{17}{2}$ wave function?

[2] (a) The Pauli matrices satisfy relations

$$\sigma_k^2 = \hat{I}, k = 1, 2, 3$$

Assuming $\hbar = 1$, what relation is implied for the spin matrices S_1, S_2 and S_3 for a spin $\frac{1}{2}$ particle?

(b) The spin one matrices are known to obey

$$S_k^3 - S_k = 0, k = 1, 2, 3$$

Give a one line explanation of the above identities.

(c) What relation will be obeyed
(i) when spin is $\frac{3}{2}$?
(ii) when spin is has arbitrary value s?

[3] The quantum states of a spin $\frac{1}{2}$ particle are represented by a two component column vector with complex elements. Let $n = (n_1, n_2, n_3)$ be a unit vector, with $n_1^2 + n_2^2 + n_3^2 = 1$. Assume that $n_3 \neq 1$. Consider the following three vectors χ_1, χ_2, χ_3 :

$$\chi_1 = \begin{pmatrix} 1+n_3 \\ n_1 - in_2 \end{pmatrix}, \qquad \chi_2 = \begin{pmatrix} 1-n_3 \\ n_1 + in_2 \end{pmatrix}, \qquad \chi_3 = \begin{pmatrix} n_1 + in_2 \\ 1 - n_3 \end{pmatrix}$$

Which of the following statements is true?

(a) All the three vectors represent the same state

(b) All the three vectors represent different states.

(c) Only two of the three vectors represent the same state and the third one represents a different state.

Give a short explanation of your answer.

[4] For a spin $\frac{1}{2}$ particle the spin wave function is given to be

$$\chi = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} \tag{30}$$

Compute average of S_y and uncertainty ΔS_y. interpret the answer you obtain.

§9 Mixed Bag

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Spin

[1] For a spin $\frac{1}{2}$ particle in arbitrary spin state, we can always find a direction along which the spin points *i.e.* the projection of spin has value has the maximum possible value $\frac{\hbar}{2}$. A similar statement for a higher spin particle will be

"Given a spin s particle in an arbitrary spin state, we can always find a direction along which the spin projection has its maximum value $s\hbar$."

Explore if this statement is true or false.

[2] For a spin half particle with spin along the unit vector $\hat{n}$ compute the average value of spin component along unit vector $\hat{m}$. Can you explain the form of the expression that you get?

UNIT-II

§10 Introduction

Let us recall the main results that are needed for a system of identical particles.The symmetrization postulate states that

- For a system of **two** identical particles with integral spin bosons the total wave function must be symmetric under simultaneous exchange of all the variables such as the space and spin variables. For a system of two identical particles of half integral spin fermions the full wave function must be anti-symmetric under a simultaneous exchange of all the variables such as the space and spin variables.

  If ξ_1, ξ_2 denote the set of all variables such as, space and spin, of two identical particles. Then the symmetrization postulate states that the total wave function $\psi(\xi_1, \xi_2)$ must be symmetric for bosons and antisymmetric for fermions under an exchange of ξ_1 and ξ_2.

$$\begin{aligned} \psi(\xi_2, \xi_1) &= +\psi(\xi_1, \xi_2) \qquad \text{(bosons)} &(31)\\ \psi(\xi_2, \xi_1) &= -\psi(\xi_1, \xi_2) \qquad \text{(fermions)} &(32)\end{aligned}$$

- The symmetrization postulate for a system of $n-$ identical particles states that the total wave function must be symmetric simultaneous under exchange of variables ξ_j and ξ_k for every pair j, k, if the particles are bosons and the relation

$$\psi(\xi_1, \cdot\xi_j \cdots, \xi_k, \cdots, \xi_n) = +\psi(\xi_1, \cdot\xi_k, \cdots, \xi_j, \cdots, \xi_n) \qquad (33)$$

  should hold for all pairs AK. Similarly, for a system of $n-$ identical fermions the total wave function must be anti-symmetric under simultaneous exchange of variables ξ_j, ξ_k for every pair j, k

$$\psi(\xi_1, \cdot\xi_j \cdots, \xi_k, \cdots, \xi_n) = -\psi(\xi_1, \cdot\xi_k, \cdots, \xi_j, \cdots, \xi_n) \qquad (34)$$

  For a system of several identical bosons, the total wave function $\Psi(\xi_1, \cdots, \xi_n)$ remains unchanged under an arbitrary permutation of $\xi_1, \cdots, \xi_n$; where as for fermions the wave function remains unchanged under an even permutation but changes sign under an odd permutations.

We now give some explanatory remarks on the symmetrization postulate.

1. The postulate is a statement about the **full** wave function of the system of identical particles under a **simultaneous exchange of all** the variables. For example, there is no constraint on the space part (or the spin part) of the wave function alone need.

2. For composite systems such those consisting of both bosons and fermions, the symmetry requirements hold for every pair of identical bosons and identical fermions separately.

3. For a system consisting of several 'particles' which themselves could be bound state of bosons and fermions the postulate applies with spin interpreted to mean the total angular momentum at rest.

4. While for a system of two particles the symmetry property is restricted to symmetry or antisymmetry alone, for a system of many identical particles theoretical considerations allow existence of a variety of possibilities under permutation of variables. These choices, known generally as 'para-statistics', do not seem to play any role for real physical systems.

Our discussion of the symmetrization postulate will be incomplete if we do not mention the spin statistics connection, presented as a postulate, has been proved by Pauli and Luders within the framework of relativistic quantum field theory under very general assumptions such as relativistic invariance, micro causality and positivity of the Hamiltonian.

§11 Problem Solving Objectives

This unit aims you to train you in the following problem solving skills.

1. You should be able to check and decide if a given properly symmetrized or antisymmetrized or not.

2. Writing correctly symmetrized,or antisymmetrized, wave function for a system of identical bosons or fermions.

3. Finding restrictions on orbital angular momentum, spin and parity of a system of identical bosons or identical fermions.

4. Understanding precise statement of the symmterization postulate

§12 Roadmap

Here give two useful statements which will be needed repeatedly in this connection.

- When we add two, equal, angular momenta j the possible resulting values are $J = 2j, 2j-1, \cdots, 0$. Of these the state with the highest value, $J = 2j$, is symmetric under an exchange of the two particles, the next one, with $2j-1$ is antisymmetric; the states being alternately symmetric and antisymmetric as J takes on the values in descending order.

- For a two particle system, the effect of an exchange of the positions of the two particles is same as the parity on the wave function in the centre of mass frame. Therefore, under an exchange of the space variables the space part of the wave function is symmetric for even ℓ and antisymmetric for odd ℓ.

§13 Exercise

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Identical Particles

[1] Write the space wave function of two electrons having definite momenta $\vec{p}_1$ and $\vec{p}_2$ for the triplet and singlet spin states.

[2] Assuming electrons in excited state of a He atom to be in hydrogen like states as specified below, write the space wave functions for the singlet and triplet states for

(a) both in 3p state with m_ℓ values 1 and -1.

(b) one electron in 2s and the other in 3s state.

What will be the total wave function, including spins, in each of the two cases?

[3] Assuming electrons in excited state of a He atom to be in hydrogen like states as specified below, write the space wave functions for the singlet and triplet states for

(a) both in 3p state with m_ℓ values 1 and -1.

(b) one electron in 2s and the other in 3s state.

What will be the total wave function, including spins, in each of the two cases?

[4] Assuming the interaction between the oxygen atoms to be spherically symmetric, what are possible values of spin of the oxygen molecule O_2. It is given that the spin of oxygen atoms is zero.

[5] For a pair of identical particles with spin s and relative angular momentum ℓ, which of the following statements is correct? Give our answer with explanation for the two case of (i)identical fermions and (ii)identical bosons .

(a) $(-1)^{\ell+S_{\rm tot}} = 1$ (b) $(-1)^{\ell+S_{tot}} = -1$ (c) none of these.

Here $S_{\rm tot}$ stands for the total spin of the two particle system.

§14 Quiz

qm-quz-20002 **Symmetrization**

Consider a system of 4 weekly interacting particles. The particles 1 and 2 are identical fermions and 3 and 4 are identical bosons. Let χ, ϕ, ψ denote possible states of the particles 1 and 2. Let f, g, h denote possible states of particles 3 and 4. The symbols $\xi_1, \xi_2, \xi_3, \xi_4$ collectively denote all the coordinates, *i.e.* space and spin variables of the four particles.

Examine the following wave functions $\Psi_1, \ldots, \Psi_5$ for the system of the four particles and state which of wave functions are admissible and which are not. If a particular wave function is not allowed for the system explain reasons why it is not allowed, giving all possible reasons

$$(a) \quad \Psi_1 = \frac{1}{24}\begin{vmatrix} \chi(\xi_1) & \phi(\xi_1) & f(\xi_1) & g(\xi_1) \\ \chi(\xi_2) & \phi(\xi_2) & f(\xi_2) & g(\xi_2) \\ \chi(\xi_3) & \phi(\xi_3) & f(\xi_3) & g(\xi_3) \\ \chi(\xi_4) & \phi(\xi_4) & f(\xi_4) & g(\xi_4) \end{vmatrix} \tag{35}$$

$$(b) \quad \Psi_2 = \frac{1}{\sqrt{2}}\begin{vmatrix} \psi(\xi_1) & \phi(\xi_1) \\ \psi(\xi_2) & \phi(\xi_2) \end{vmatrix} f(\xi_3)f(\xi_4) \tag{36}$$

$$(c) \quad \Psi_3 = \frac{1}{\sqrt{2}}\chi(\xi_1)\chi(\xi_2)\begin{vmatrix} f(\xi_3) & g(\xi_3) \\ f(\xi_4) & g(\xi_4) \end{vmatrix} \tag{37}$$

$$(d) \quad \Psi_4 = (\psi(\xi_1)\phi(\xi_2) + \psi(\xi_2)\phi(\xi_3))h(\xi_3)h(\xi_4) \tag{38}$$

$$(e) \quad \Psi_5 = \frac{1}{\sqrt{2}}\big(\chi(\xi_1)\phi(\xi_2)f(\xi_3)g(\xi_4) - \chi(\xi_2)\phi(\xi_1)f(\xi_4)g(\xi_3)\big) \tag{39}$$

§15 Exercise

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Identical Interacting Particles

[1] Assuming that two identical spin zero bosons interact via a spherically symmetric potential, show that the system at rest cannot have odd angular momentum and odd parity, it must can only have even spin parity $0^+, 2^+, 4^+ ....$

[2] The states occupied by the two electrons are specified in the second column of the table given below. Write properly symmetrized total wave function for two electron system in the last column of the table given below. (Assume $\vec{p} \neq \vec{q}$.)

SN	State Description	Total wave function
1	One of the two electrons has momentum $\vec{p}$ and spin up; The other electron has momentum $\vec{q}$ and spin down.	
2	The two electrons are in a spin triplet state and have momenta $\vec{p}, \vec{q}$.	
3	The two electrons have momenta $\vec{p}, \vec{q}$ and spin state is singlet.	
4	The two electrons have momenta $\vec{p}, \vec{q}$ and both electrons have spin up.	
5	Both electrons have momentum $\vec{p}$.	

⊘ Use notation α, β to denote, respectively, the spin up and spin down states of a spin half particles. So for example, the spin wave function for a system of two particles (need not be identical) with the first particle having spin up and the second particle having spin down will be $\alpha(1)\beta(2)$.

[3] Neglecting the electrostatic repulsion of electrons one can assume that the two electrons in a He atom occupy hydrogen like 'orbits' The quantum numbers of the two electrons are specified in the second column of the table given below. Write

properly symmetrized total wave function for two electron system in the last column of the table given below.

Here n, n', are not the principal quantum numbers, and will be used to denote the radial quantum numbers of the two electrons. We will use N, N' to denote the principle quantum numbers. (Note principal quantum number of an electron will then be $N = (n + \ell + 1)$ etc). The symbols $\ell, m; \ell', m'; J, M; J', M'$ etc have their usual meanings and, if needed, spin states are specified explicitly.

You may use the notation $R_{n\ell}(r)Y_{\ell m}(\theta, \phi)$ for wave function of an electron in hydrogen like orbit with quantum numbers n, ℓ, m.

Use notation α, β to denote, respectively, the spin up and spin down states of a spin half particles. So for example, the spin wave function for a system of two particles (need not be identical) with the first particle having spin up and the second particle having spin down will be $\alpha(1)\beta(2)$.

SN	State Description	Unique State? (Y/N)	Total wave function
1	$N_1 = 1, \ell_1 = 0, m_1 = 0;$ $N_2 = 2, \ell = 1, m_2 = 1$ Spins $\uparrow, \downarrow$		
2	$N_1 = 1, \ell_1 = 0, m_1 = 0;$ $N_2 = 2, \ell_2 = 1, m_2 = -1;$		
3	$N_1 = 1, N_2 = 1$		
5	The two electrons are in a spin triplet state and have momenta $\vec{p}, \vec{q}$		
6	$n_1 = n_2 = 1, \ell_1 = \ell_2 = 1;$ $L = 1, M = 0$		
7	$n_1 = n_2 = 1, \ell_1 = \ell_2 = 1$ $L = 2, M = 0$		
8	$n_1 = 2, \ell_1 = 1, J_1 = 1/2$ $n_2 = 2, \ell_2 = 1, J_2 = 3/2$ $M_1 = M_2 = 1/2$		
9	$n_1 = n_2 = 1, \ell_1 = \ell_2 = 1$ $J = 0, M = 0$		
10	Spin state is singlet $n_1 = n_2 = 1, \ell_1 = \ell_2 = 1$ Spin State is triplet		

Here L is the total orbital angular momentum for the two electrons.

§16 Quiz

□ Symmetrization Postulate

Consider a system of two weekly interacting spin one bosons. Use the following notation to answer the questions given below. Possible spin states are denoted by α, β, γ for spin projections $\hbar, 0, -\hbar$ and the spatial 'orbits' available are described by wave functions ψ and ϕ.

Write all possible properly symmetrized total wave functions when the bosons occupy states as described in the second column. Write the total wave function in the last column. The entries in the first row show a sample answer.

S.No.	Particulars	Wave function
Sample	Both bosons are in state u and one has spin 1 and the other one has S_z value 0	$\frac{1}{\sqrt{2}}u(\vec{r}_1)u(\vec{r}_2)\big(\alpha(1)\beta(2)+\alpha(2)\beta(1)\big)$
1	Total spin of bosons is 2; Write allowed space wave function(s).	
2	Total spin of bosons is 1; Write allowed space wave function(s)	
3	All the three bosons have spin up; Write allowed space wave function(s)	
4	The spin wave function is given to be $\frac{1}{\sqrt{2}}(\alpha(1)\beta(2)-\alpha(2)\beta(1)\big)$, Write space wave function(s)	
5	The space wave function is $u(\vec{r}_1)v(\vec{r}_2)-u(\vec{r}_2)v(\vec{r}_1)$ Write allowed spin wave function(s)	
6	The space wave function is $u(\vec{r}_1)v(\vec{r}_2)+u(\vec{r}_2)v(\vec{r}_1)$ Write allowed spin wave function(s)	

Write all answers if more than one answer is possible.

§17 Mixed Bag

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Wave functions

[1] Show that for a system of two identical particles having spin s, the ratio of the number of states, symmetric under exchange of spins, to the number of the antisymmetric states is given by $\frac{(s+1)}{s}$.

[2] Consider two particles, A, B each having spin 1. Construct all possible states $|SM\rangle$ with definite values of S^2 and S_z. Use the notation $|Am_1\rangle|Bm_2\rangle$ to represent the states of the two particles having definite $z-$ projections m_1, m_2 of spin. Verify that the states are symmetric under exchange of m_1 and m_2 when total spin is $S = 2, 0$ and antisymmetric for $S = 1$.

[3] Assuming that two identical spin zero bosons interact via a spherically symmetric potential, show that the system at rest cannot have odd angular momentum and odd parity, it must can only have even spin parity $0^+, 2^+, 4^+ ....$

[4] Show that the operators

$$\Lambda_\ell^+ = \frac{\ell + 1 + \vec{\sigma} \cdot \vec{L}}{2\ell + 1}, \qquad \Lambda_\ell^- = \frac{\ell - \vec{\sigma} \cdot \vec{L}}{2\ell + 1}$$

are projection operators onto states $j = \ell \pm \frac{1}{2}$, respectively, in the subspace of orbital angular momentum ℓ. (Here we are considering a single spin $\frac{1}{2}$ particle.)

Gottfried [?]

[5] Two spin half fermions, mass μ and positions $\vec{r}_1, \vec{r}_2$, interact via anisotropic harmonic oscillator potential

$$V(\vec{r}_1, \vec{r}_2) = \frac{1}{2}k_1(x_1 - x_2)^2 + \frac{1}{2}k_2(y_1 - y_2)^2 + \frac{1}{2}k_3(z_1 - z_2)^2$$

What are the energy levels of the system? Specify allowed values of the total spin for each energy level. Compare your answer with the case of two nonidentical spin half fermions.

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