Phy 523 PARTICLE PHYSICS SOLUTIONS PROBLEM SHEET I

1.Consider a particle A (mass M_A) decaying to two particles B,C (masses M_B, M_C). Find the momentum of particle B in the (a) rest frame of A and (b) in a frame in which the momentum of A is $\vec{P_A} = (0, 0, P_A)$ and the momentum of B makes an angle θ with respect otA.

Solution:

(a) In the rest frame of A the four momenta of the three particles can be written as $P_A^\mu = (m_A, \vec{0}), P_B^\mu = (E_B(r), \vec{P}_B(r)), P_C^\mu = (E_C(r), \vec{P}_C(r))$ Conservation of four momentum leads to

$$P_A^\mu = P_B^\mu + P_C^\mu$$

or

$$P_A^\mu - P_B^\mu = P_C^\mu$$

Squaring

$$(P_A^\mu - P_B^\mu)(P_{A\mu} - P_{B\mu}) = P_C^\mu P_{C\mu} = m_C^2$$
$$= m_A^2 + m_B^2 - 2P_A^\mu P_{B\mu}$$

We have

$$P_A^\mu P_{B\mu} = m_A E_B(r)$$

Thus

$$m_A^2 + m_B^2 - 2m_A E_B(r) = m_C^2$$

or

$$E_B(r) = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A}$$

the momeentum of B is

$$|\vec{P}_B(r)| = (E_B^2(r) - m_B^2)^{1/2} = \frac{(m_A^4 + m_B^4 + m_C^4 - 2m_A m_B - 2m_A m_C - 2m_B m_C)^{1/2}}{2m_A}$$

$E_B(r), |\vec{P}_B(r)|$ refer to the energy and momentum of B at rest.

(b) Here choose

$$P_A^\mu = (E_A, 0, 0, |\vec{P}_A|), P_B^\mu = (E_B, |\vec{P}_B| sin(\theta) cos(\phi), |\vec{P}_B| sin(\theta) sin(\phi), |\vec{P}_B| cos(\theta))$$

$$P_C^\mu = (E_C, \vec{P}_A - \vec{P}_B)$$

Again

$$(P_A^\mu - P_B^\mu)(P_{A\mu} - P_{B\mu}) = P_C^\mu P_{C\mu} = m_C^2$$

or

$$m_A^2 + m_B^2 - 2P_A^\mu P_{B\mu} = m_C^2$$

$$m_A^2 + m_B^2 - m_C^2 = 2P_A^\mu P_{B\mu} = 2(E_A E_B - |\vec{P}_A||\vec{P}_B|cos(\theta))$$

$$\frac{m_A^2 + m_B^2 - m_C^2}{2} = E_A(m_B^2 + |\vec{P}_B|^2)^{1/2} - |\vec{P}_A||\vec{P}_B|cos(\theta) = m_A E_B(r)$$

$$(M_A E_B(r) + |\vec{P}_A||\vec{P}_B|cos(\theta))^2 = E_A^2(m_B^2 + |\vec{P}_B|^2)$$

$$M_A^2 E_B(r)^2 + |\vec{P}_A|^2|\vec{P}_B|^2 cos(\theta)^2 + 2M_A E_B(r)|\vec{P}_A||\vec{P}_B|cos(\theta) = E_A^2 m_B^2 + E_A^2|\vec{P}_B|^2$$

$$(E_A^2 - |\vec{P}_A|^2 cos^2(\theta)|\vec{P}_B| - 2m_A|\vec{P}_A|E_B(r)cos(\theta)|\vec{P}_B| + E_A^2 m_B^2 - m_A^2 E_B(r)^2 = 0$$

solving for $|\vec{P}_B|$ we get

$$|\vec{P}_B| = \frac{2m_A|\vec{P}_A|E_B(r)cos(\theta)}{2(E^2 - |\vec{P}_A|^2 cos^2(\theta))}$$

$$\pm \frac{(4m_A^2|\vec{P}_A|^2 E_B(r)^2 cos^2(\theta) - 4(E_A^2 m_B^2 - m_A^2 E_B^2(r))(E_A^2 - |\vec{P}_A|^2 cos^2(\theta))^{1/2}}{2(E^2 - |\vec{P}_A|^2 cos^2(\theta))}$$

The $-$ve sign would give $|\vec{P}_B| < 0$ as $|\vec{P}_A| \to 0$ and so we chose the +-ve sign. Simplifying the term inside the squareroot we get

$$4m_A^2|\vec{P}_A|^2 E_B(r)^2 cos^2(\theta) - 4(E_A^2 m_B^2 - m_A^2 E_B^2(r))(E_A^2 - |\vec{P}_A|^2 cos^2(\theta))$$

$$= 4m_A^2 E_B(r)^2(|\vec{P}_A|^2 cos^2(\theta) + (E_A^2 - |\vec{P}_A|^2 cos^2(\theta)) - 4E_A^2 m_B^2(E_A^2 - |\vec{P}_A|^2 cos^2(\theta))$$

$$= 4m_A^2 E_B(r)^2 E_A^2 - 4E_A^2 m_B^2(E_A^2 - |\vec{P}_A|^2 cos^2(\theta))$$

$$= 4E_A^2(m_A^2 E_B(r)^2 - E_A^2 m_B^2) + 4E_A^2 m_B^2|\vec{P}_A|^2 cos^2(\theta)$$

$$= 4E_A^2(m_A^2 E_B(r)^2 - (|\vec{P}_A|^2 + m_A^2)m_B^2) + 4E_A^2 m_B^2|\vec{P}_A|^2 cos^2(\theta))$$

$$= 4E_A^2(m_A^2(E_B(r)^2 - m_B^2) - |\vec{P}_A|^2 m_B^2 + m_B^2|\vec{P}_A|^2 cos^2(\theta))$$

$$= 4E_A^2(m_A^2|\vec{P}_B(r)|^2 - m_B^2|\vec{P}_A|^2 sin^2(\theta)$$

Thus

$$|\vec{P}_B| = \frac{2m_A|\vec{P}_A|E_B(r)cos(\theta) + 2E_A(m_A^2|\vec{P}_B(r)|^2 - m_B^2|\vec{P}_A|^2 sin^2(\theta))^{1/2}}{2(E_A^2 - |\vec{P}_A|^2 cos^2(\theta))}$$

2. Consider a particle A (mass M_A) decay to three particles B,C and D (masses M_B, M_C and M_D). Find the maximum and the minimum energy range for C in the rest frame of A.

Solution:

Conservation of energy and momentum demand

$$M_A = E_A + E_B + E_C = (m_B^2 + |\vec{P}_B|^2)^{1/2} + (m_C^2 + |\vec{P}_C|^2)^{1/2}(m_D^2 + |\vec{P}_D|^2)^{1/2}$$

$$\vec{P}_A = 0 = \vec{P}_B + \vec{P}_C + \vec{P}_D$$

We have to maximse E_C subject to the above conditions. Using the method of Langrange multipliers we consider the function

$$F(\vec{P}_B, \vec{P}_C, \vec{P}_D) = E_C - \lambda(E_B + E_C + E_D) - \sigma^i(P_B^i + P_C^i - P_D^i$$

λ and $\sigma^i \quad i = 1, 2, 3$ are the Langrange multipliers. We can now treat $\vec{P}_A, \vec{P}_B, \vec{P}_C$ as independent variables. For a stationary solution

$$\frac{\partial F(\vec{P}_A, \vec{P}_B, \vec{P}_C)}{\partial P_B^i} = -\lambda \frac{P_B^i}{E_B} - \sigma^i = 0$$

$$\frac{\partial F(\vec{P}_A, \vec{P}_B, \vec{P}_C)}{\partial P_C^i} = 1 - \lambda \frac{P_C^i}{E_C} - \sigma^i = 0$$

$$\frac{\partial F(\vec{P}_A, \vec{P}_B, \vec{P}_C)}{\partial P_D^i} = -\lambda \frac{P_D^i}{E_D} - \sigma^i = 0$$

The above equations imply $\frac{P_B^i}{E_B} = \frac{P_D^i}{E_D} = \frac{\sigma^i}{\lambda}$. This means the velocity of B and D are the same and one can consider them as moving together. So we can treat the problem as a two body decay with A decaying to $(B + D)$ with a mass of $m_B + m_D$ and C. Thus

$$E_C(max) = \frac{m_A^2 + m_C^2 - (m_B + m_D)^2}{2m_A}.$$

The minimum value of E_C is m_C when it is at rest with B and C taking away the energy $m_A - m_C$ and $\vec{P}_B + \vec{P}_B = 0$.

3. Consider the scattering $A+B \to C+D$. Masses are (M_A, M_B, M_C, M_D) respectively with $M_C + M_D > M_A + M_D$. Find the minimum energy needed for the particle A in order this reaction occurs (a) in the rest frame of B (b) in the centre of mass frame.

Solution: (a) We have the conservation law

$$P_A^\mu + P_B^\mu = P_C^\mu + P_D^\mu$$

Note that C and D can not be produced at rest in the rest frame of B as the initial three momentum of B is not zero and hence momentum of the final state can not be zero. We refer to this frame as the 'lab' frame In this frame we write

$$\begin{aligned} P_A^\mu = (E_A(l), 0, 0, P_A); P_B^\mu = (m_B, 0, 0, 0) \\ (P_{A\mu} + P_{B\mu})(P_A^\mu + P_B^\mu) = m_A^2 + m_B^2 + 2P_{A\mu}.P_B^\mu \\ = m_A^2 + m_B^2 + 2m_B E_A(l) \end{aligned}$$

This also equals $(P_{C\mu} + P_{D\mu})(P_C^\mu + P_D^\mu) = m_C^2 + m_D^2 + 2P_{C\mu}P_D^\mu$. We therefore need to find the minimum value of $2P_{C\mu}P_D^\mu$. Since it is a Lorentz scalar, this can be evaluated in any frame. Choose a frame in which C is at rest. In this frame $2P_{C\mu}P_D^\mu = m_C E_D$(rest frame of C). The minimum of energy of D is m_D and thus the minimum value is $m_C m_D$. In this configuration both C and D travel together. Thus

$$m_A^2 + m_B^2 + 2m_B E_A(l) = m_C^2 + m_D^2 + m_C m_D = (m_C + m_D)^2$$

or

$$E_A(l) = \frac{(m_C + m_D)^2 - m_A^2 - m_B^2}{2m_B}$$

(b)In the centre of mass frame, $\vec{P}_A = -\vec{P}_B; \vec{P}_C = -\vec{P}_D$ and the minimum energy configuration is when C and D are produced at rest. This is allowed as the initial total momentum is zero. We can write for this configuration

$$(P_C^\mu + P_D^\mu) = (m_C + m_D, 0, 0, 0); P_A^\mu = (E_A(c.m), 0, 0, P_A(c.m)); P_B^\mu = (E_B(c.m), 0, 0, -P_A(c.m))$$

Also

$$P_B^\mu = (P_C^\mu + P_D^\mu) - P_A^\mu$$

Squaring we get

$$m_B^2 = (m_C + m_D)^2 + m_A^2 - 2(m_C + m_D)E_A(c.m)$$

or

$$E_A(c.m) = \frac{(m_C + m_D)^2 - m_A^2 - m_B^2}{2(m_C + m_D)}$$

4. Consider the decay $\pi^0 \to \gamma + \gamma$ (mass of pion is M_π). Assume that the decay is isotropic in the rest frame of π^0. Now consider a frame π^0 is moving with energy E. Show that the energy distribution of the photons in this frame is given by

$$\frac{dN(E_\gamma)}{dE_\gamma} = constant$$

where $N(E_\gamma)$ is the number of photons emitted as a funcion of the energy of the photon E_γ. Further show that $E(1-\beta) < E_\gamma < E(1+\beta)$ where $\beta = (E^2 - M_\pi^2)^{1/2}/E$

Solution:

Let the pion momentum be along the third direction. In the rest frame of pion let one of the photon have the four momenta $P^\mu_{\gamma 1} = P^\mu_{\gamma 1} = m_\pi/2(1, sin(\theta), 0, cos(\theta))$. The other photon would have $P_{\gamma 2} = m_\pi/2(1, -sin(\theta), 0, -cos(\theta))$. We can transform this four vector to the frame in which pion has energy E. Its three momentum vector is $(0, 0, p = (E^2 - m_\pi^2)^{1/2})$ If E_1' is the energy of the photon 1 in the frame in which pion is moving,we have

$$E_1' = \gamma(\frac{m_\pi}{2} + \beta\frac{m_\pi}{2}cos(\theta) = \frac{E}{2}(1 + \beta cos(\theta)$$

as $\gamma = E/m_\pi$. thus

$$\frac{E}{2}(1-\beta) < E_1' < \frac{E}{2}(1+\beta)$$

In the rest frame of the pion the distribution of the photon is isotropic.

$$\frac{dN}{dcos(\theta)} = K$$

where K is a constant. we also have $\frac{dE_1'}{dcos(\theta)} = \frac{E\beta}{2}$ Thus we write

$$\frac{dN(E_1')}{dE_1'} = \frac{dN}{dcos(\theta)}\frac{dcos(\theta)}{dE_1'} = \frac{2}{E\beta}\frac{dN}{dcos(\theta)}$$

$$= K\frac{2}{E\beta} = constant.$$

5.(a) The life time of a particle in natural units ($\hbar = c = 1$) is $1Mev^{-1}$. Find the lifetime in seconds.

(b) the cross section of scattering in a process is $1Mev^{-2}$. Find the cross section in cm^2

Solution:

$$\tau = \frac{\hbar}{E} = \frac{\hbar}{MeV} = \frac{1.055 \times 10^{-34}}{1.602 \times 10^{-13}} = 6.58 \times 10^{-22}$$

$$\sigma = \frac{\hbar^2 c^2}{MeV)^2} = \left(\frac{1.05 \times 10^{-34} 3 \times 10^8}{1.602 \times 10^{-18}}\right) m^2$$

$$= 3.9 \times 10^{-22} cm^2$$