21.8.2017

P1/Q13/ §§7.6

§§7.6 Show that
Q13

$$\int_0^\infty \frac{x^{p-1}\,dx}{x^3+b^3} = \frac{\pi b^{p-3}}{3\sin\pi p}\left(1+2\cos(2\pi p/3)\right)$$

Taking the definition of z^p as

$$z^p = r^p e^{ip\theta} \quad 0<\theta<2\pi, \quad \text{------- (1)}$$

We set up the integral of $f(z) = z^{p-1}/(z^3+b^3)$ around the double circles contour of fig 1.

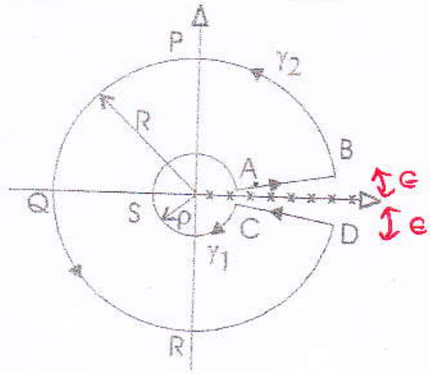

Fig. 1 Contour Γ.

$$\oint_\Gamma \frac{z^{p-1}}{z^3+b^3}dz = \int_{AB} f(z)\,dz + \int_{BPQRD} f(z)\,dz + \int_{DC} f(z)\,dz + \int_{CSA} f(z)\,dz$$

We will take limits $\rho\to 0$, $R\to\infty$. In this limit the integrals along circular are vanish. Therefore

$$\oint_\Gamma \frac{z^{p-1}}{z^3+b^3}dz = \int_{AB}\frac{z^{p-1}}{z^3+b^3}dz - \int_{CD}\frac{z^{p-1}}{z^3+b^3}dz \quad \text{--- (2)}$$

Next set up the two integrals in the right hand side of the above equation.

AB: $z = re^{i\epsilon}$ $\rho<r<R$

$dz = dr\,e^{i\epsilon}$

$$f(z) = \frac{r^{p-1}e^{i(p-1)}}{r^3+b^3}$$

CD: $z = re^{i(2\pi-\epsilon)}$

$dz = dr\,e^{i(2\pi-\epsilon)}$

$$f(z) = \frac{r^{p-1}e^{i(2\pi-\epsilon)(p-1)}}{r^3+b^3}.$$

Therefore Eq (2) takes the form, in the limit $\epsilon \to 0$,

$$\oint_{\Gamma} f(z)\,dz = \int_{\rho}^{R} \frac{r^{p-1}(1-e^{2\pi i p})}{r^3+b^3}\,dr$$

$$\therefore \quad = -2i \sin \pi p\, e^{ip\pi} \int_{\rho}^{R} \frac{r^{p-1}\,dr}{r^3+b^3}$$

Hence in the limit $\rho \to 0$, $R \to \infty$,

$$\int_0^\infty \frac{x^{p-1}}{x^3+b^3}\,dx = \frac{-e^{-ip\pi}}{2i \sin \pi p} \oint_{\Gamma} \frac{z^{p-1}}{z^3+b^3}\,dz \quad \text{----}(3)$$

The integral in the right hand side of (3) can now be computed using the residue theorem. The function $f(z)$ has poles at $z = b\,e^{i\pi/3}$, $b\,e^{i\pi/3+2\pi/3}$, $b\,e^{i\pi/3+4\pi/3}$

Let ξ denote any one of these points, $\xi^3 = -b^3$ and

$$\text{Res}\{f(z)\}_{z=\xi} = \lim_{z\to\xi} (z-\xi)\left(\frac{z^{p-1}}{z^3+b^3}\right)$$

$$= \left.\frac{z^{p-1}}{3z^2}\right|_{z=\xi}$$

$$= \left.\frac{z^p}{3z^3}\right|_{z=\xi} = \left(\frac{\xi^p}{-3b^3}\right) \quad \because \xi^3 = -b^3$$

While computing the residue $\arg \xi$ must be kept in the range $(0, 2\pi)$ (See (1))

$$\therefore \text{sum of all residues} = \frac{-b^p}{3b^3}\left[e^{ip\pi/3} + e^{ip(\pi/3+2\pi/3)} + e^{ip(\pi/3+4\pi/3)}\right]$$

$\therefore$ Sum of all residues P3/Q13/§§7.6

$$= -\frac{b^p}{3b^3} e^{ip\pi/3}\left[1 + e^{2\pi i p/3} + e^{4\pi i p/3}\right]$$

$$= \left(\frac{-b^p}{3b^3}\right) e^{ip\pi/3} e^{+2\pi i p/3}\left[e^{-2\pi i p/3} + 1 + e^{2\pi i p/3}\right]$$

$$= \left(\frac{-e^{\pi i p}}{3b^3}\right)\left(1 + 2\cos(2\pi p/3)\right) b^p \quad \text{-----} \quad (4)$$

Using the sum of all residues in (3) we get-

$$\int_0^\infty \frac{x^{p-1}}{x^3+b^3}\,dx = 2\pi i \times \left(\frac{-e^{i\pi p}}{3b^3}\right)\left(1 + 2\cos\left(\frac{2p\pi}{3}\right)\right) \times b^p$$

$$\times \left(\frac{-e^{-i\pi p}}{2i\sin \pi p}\right)$$

$$= \left(\frac{\pi b^{p-3}}{\sin \pi p}\right)\left(1 + 2\cos\left(\frac{2\pi p}{3}\right)\right)$$