9.12.2017 Q13

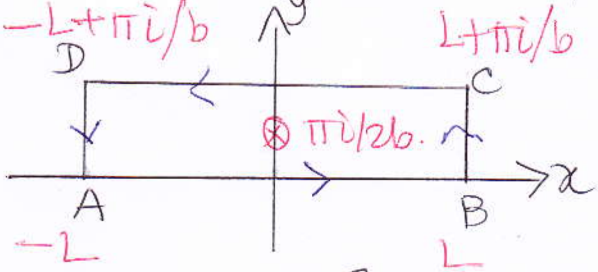
§87.9

Compute the integral $\int_0^\infty x^2 \frac{\cosh ax}{\cosh bx}\, dx$, $(b > a > 0)$ using the method of contour integration.

We first write the given integral as

$$\int_0^\infty \frac{x^2 \cosh ax}{\cosh bx}\, dx = \int_{-\infty}^{\infty} \frac{x^2 e^{ax}}{\cosh bx}\, dx \quad \text{------}$$

Consider integral of $f(z) = \frac{z^2 e^{az}}{\cosh bz}$ around closed rectangular contour Γ shown in figure. The corners ABCD of the rectangle are taken at $-L$, L, $L+\pi i$ and $-L+\pi i$. In the limit $L \to \infty$ the integrals along the vertical lines vanish and we get



Contour Γ

$$\lim \oint_\Gamma f(z)\,dz = \int_{AB} f(z)\,dz + \int_{CD} f(z)\,dz$$

$$= \int_{AB} f(z)\,dz - \int_{DC} f(z)\,dz$$

$$= \int_{-\infty}^{\infty} \frac{x^2 e^{ax}}{\cosh bx}\, dx - \int_{-\infty}^{\infty} \frac{(x+\pi i/b)^2 e^{ax+\pi i a/b}}{\cosh(\pi i + bx)}$$

$$= \int_{-\infty}^{\infty} \frac{x^2 e^{ax}}{\cosh bx}\, dx + \int_{-\infty}^{\infty} \frac{e^{\pi i a/b} (x+\pi i/b)^2 e^{ax}}{\cosh bx}\, dx \quad \text{--- (1)}$$

$$= (1+e^{i\pi a/b})\int_{-\infty}^{\infty}\frac{x^2e^{ax}}{\cosh bx}\,dx + \frac{2\pi i}{b}e^{i\pi a/b}\int_{-\infty}^{\infty}\frac{x\,e^{ax}}{\cosh bx}$$

$$+\left(\frac{\pi i}{b}\right)^2\int_{-\infty}^{\infty}\frac{e^{ax}}{\cosh bx}\,dx \times e^{\pi i a/b}. \qquad \text{---- (2)}$$

The contour Γ encloses one simple pole of $f(z)$ at $z = \pi i/(2b)$. Hence

$$\oint_\Gamma f(z)\,dz = (2\pi i)\,\mathrm{Res}\left\{\frac{z^2e^{az}}{\cosh bz}\right\}_{z=i\pi/(2b)}$$

$$= 2\pi i \lim_{z=i\pi/2b} z^2e^{az}\,\frac{(z-i\pi/2b)}{\cosh bz}$$

$$= \left(\frac{-\pi^2}{4b^2}\right)e^{i\pi a/2b} \times \frac{1}{b.\underbrace{\sinh i\pi/2}_{i}} \times (2\pi i)$$

$$= \left(\frac{i\pi^2}{4b^3}\right)e^{i\pi a/2b}. \times (2\pi i) = -\frac{\pi^3}{2b^3}e^{i\pi a/2b} \quad \text{--- (3)}$$

Substituting (3) for the left hand side in (2) and multiplying by $e^{-i\pi a/b}$ gives.

$$\left(1+e^{-i\pi a/b}\right)\int_{-\infty}^{\infty}\frac{x^2e^{ax}\,dx}{\cosh bx} + \frac{2\pi i}{b}\int_{-\infty}^{\infty}\frac{x\,e^{ax}}{\cosh bx}\,dx$$

$$-\frac{\pi^2}{b^2}\int_0^{\infty}\frac{e^{ax}}{\cosh(bx)}\,dx = -\frac{\pi^3}{2b^3}.e^{-i\pi a/2b}. \quad \text{--- (4)}$$

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Using I_n to denote $\int_{-\infty}^{\infty} \frac{x^n e^{ax}}{\cosh bx} dx$ we get, taking imaginary part of L.H.S. & RHS. of Eq(4)

$$-\sin\left(\frac{\pi a}{b}\right) I_2 + \left(\frac{2\pi}{b}\right) I_1 = +\frac{\pi^3}{2b^3} \sin\left(\frac{\pi a}{2b}\right) \quad \text{---(5)}$$

Using

$$I_1 = \int_{-\infty}^{\infty} \frac{x e^{ax}}{\cosh bx} dx = 2\int_0^{\infty} \frac{x \sinh ax}{\cosh bx} dx$$

$$= 2 \times \frac{\pi^2}{4b^2} \sin\left(\frac{\pi a}{2b}\right) \sec^2\left(\frac{\pi a}{2b}\right)$$

see problem Q14 §§7.9

Substituting value of I_2 in (5) gives

$$+2\sin\left(\cancel{\frac{\pi a}{2b}}\right) \cos\left(\frac{\pi a}{2b}\right) I_2 = \left(\frac{2\pi}{b}\right) \times \frac{\pi^2}{2b^2} \sin\left(\cancel{\frac{\pi a}{2b}}\right) \sec^2\frac{\pi a}{2b}$$

$$- \frac{\pi^3}{2b^3} \sin\left(\cancel{\frac{\pi a}{2b}}\right)$$

We therefore get

$$I_2 = \left(\frac{\pi^3}{2b^3}\right) \sec^3\left(\frac{\pi a}{2b}\right) - \frac{\pi}{4b^3} \sec\left(\frac{\pi a}{2b}\right)$$

$$= \frac{\pi^3}{4b^3} \left(2\sec^3\left(\frac{\pi a}{2b}\right) - \sec\frac{\pi a}{2b}\right)$$

and $\int_0^{\infty} \frac{x^2 \cosh ax}{\cosh bx} dx = \frac{1}{2} I_2 = \frac{\pi^3}{8b^3}\left(2\sec^3\left(\frac{\pi a}{2b}\right) - \sec\frac{\pi a}{2b}\right)$