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§§7.7 Q1 Integrate $\int_0^\infty \frac{\log x}{x^2+1}\,dx$ using the method of contour integration.

For this integral we take $\log z$ to be defined as principal value

$\mathrm{Log}\, z = \ln r + i\theta, \quad -\pi < \theta < \pi$

and integrate $f(z) = \mathrm{Log}\, z/(z^2+1)$ along contour Γ of figure 1. The limits $R \to \infty$, $\epsilon \to 0$ will be taken after setting up the integral.

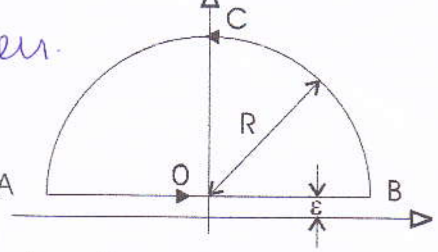

Fig. 1 Contour Γ

$$\oint_\Gamma \frac{\mathrm{Log}\, z}{(z^2+1)}\,dz = \int_{AO} f(z)\,dz + \int_{OB} f(z)\,dz + \int_{BCA} f(z)\,dz$$

The integral $\int_{BCA} f(z)\,dz$ goes to

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The left hand side is computed the residue theorem

$$\oint \frac{\mathrm{Log}\, z}{z^2+1}\,dz = 2\pi i \times \mathrm{Res}\left\{\frac{\mathrm{Log}\, z}{(z^2+1)}\right\}_{z=i}$$

$$= 2\pi i \times \lim_{z\to i}\left[(z-i)\frac{\mathrm{Log}\, z}{(z^2+1)}\right]$$

$$= 2\pi i \times \frac{\ln 1 + i\pi/2}{2i}$$

$$= +\frac{i\pi^2}{2}$$

$\therefore$ Eq (1) gives

$$\frac{i\pi^2}{2} = 2\int_0^\infty \frac{\log r\, dr}{r^2+1} + i\pi\int_0^\infty \frac{dr}{r^2+1}$$

Equating real and imaginary parts we get

$$\int_0^\infty \frac{\log x\, dx}{x^2+1} = 0 \qquad \int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$$

Note that for purpose of using contour Γ of Fig 1 the branch cut could be taken along any ray in the lower half plane.