QM-Lecture Notes 6.4
General Principles of Quantum Mechanics*
Simultaneous measurement

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§1 Compatible Observables

Let A and B be two dynamical variables, $\hat{A}, \hat{B}$ be the corresponding operators, $\alpha_j, j = 1, 2, \ldots$ and $\beta_k, k = 1, 2, \ldots$ be their eigenvalues.

Let us assume that A and B can be measured simultaneously. This means there are states in which these variables have definite values $\alpha_j, \beta_k, j, k = 1, 2, \ldots$.The corresponding vectors $|\alpha_j, \beta_k\rangle$ must then be simultaneous eigenvectors of the the two operators.

$$\hat{A}|\alpha_j, \beta_k\rangle = \alpha_j|\alpha_j, \beta_k\rangle, \qquad \hat{B}|\alpha_j, \beta_k\rangle = \beta_k|\alpha_j, \beta_k\rangle \tag{1}$$

In order that the probability of getting pair of values α_j, β_k for all pairs j, k be given by the postulate III, it should be possible to write an arbitrary vector $|\psi\rangle$ as linear combination of these vectors $\mathscr{B} = \{|\alpha_j, \beta_k\rangle | j = 1, 2, ..., k = 1, 2, ...\}$ and these states must form a basis.

Now it is easy to show that the action of $\hat{A}\hat{B} - \hat{B}\hat{A}$ on each of these vectors in the set $\mathscr{B}$ is zero. In fact

$$\begin{aligned}
\big(\hat{A}\hat{B} - \hat{B}\hat{A}\big)|\alpha_j, \beta_k\rangle &= \hat{A}\hat{B}|\alpha_j, \beta_k\rangle - \hat{B}\hat{A}|\alpha_j, \beta_k\rangle & (2)\\
&= \hat{A}\beta_k|\alpha_j, \beta_k\rangle - \hat{B}\alpha_j|\alpha_j, \beta_k\rangle & (3)\\
&= \alpha_j\beta_k|\alpha_j, \beta_k\rangle - \beta_k\alpha_j|\alpha_j, \beta_k\rangle & (4)\\
&= 0. & (5)
\end{aligned}$$

Thus we have proved that the action of commutator $[\hat{A}, \hat{B}]$ on every element of basis $\mathscr{B}$ results in zero. This implies that $[\hat{A}, \hat{B}] = 0$ and the two operators $\hat{A}, \hat{B}$ must commute.

Conversely, if two hermitian operators commute, one can select a basis of orthonormal vectors which are simultaneous eigenvectors of the two operators.

The above considerations generalize to several dynamical variables.

A set of operators $\{\hat{A}_k, k = 1, 2, \ldots\}$ is called *commuting set* if every pair of operators $\hat{A}_\ell, \hat{A}_m$ commute, *i.e.*

$$[\hat{A}_\ell, \hat{A}_m] = 0 \quad \text{for all pairs} \ell, m. \tag{6}$$

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A set of dynamical variables $\{A_k, k = 1, 2, \ldots\}$ is called a compatible set if the corresponding set of operators $\{\hat{A}_k, k = 1, 2, \ldots\}$ is a commuting set of operators. It then follows that

> REMEMBER
> "A set of dynamical variables an be can be measured simultaneously if and only if they commute pairwise. In other words they should form a compatible set."

§2 Functions of Operators

Let $\widehat{X}$ be an operator which has eigenvalues and eigenvectors $\{\lambda_k, |u_k\rangle | k = 1, 2, ...\}$. Let us further assume that the span of eigenvectors of $\widehat{X}$ is entire vector space. Then a function $\widehat{F}(X)$ of the operator $\widehat{X}$ is defined by specifying its action on the basis formed by the eigenvectors $\mathscr{B} = \{|u_k\rangle | k = 1, 2, ...\}$

$$\widehat{F}(X)|u_k\rangle \equiv F(\lambda_k)|u_k\rangle, \quad k = 1, 2, \ldots. \tag{7}$$

The action of the function $\widehat{F}(X)$ on an arbitrary vector $|\psi\rangle$ is obtained, as usual, by expanding the vector $|\psi\rangle$ in the basis $\mathscr{B}$:

$$|\psi\rangle = \sum_k c_k |u_k\rangle \tag{8}$$

$$\widehat{F}(X)|\psi\rangle = \sum c_k F(\lambda_k)|u_k\rangle. \tag{9}$$

As the hermitian operators and unitary operators have a complete set of orthonormal eigenvectors, their functions are defined by the method outlined above.

It is a simple exercise to show that if an operator $\widehat{Y}$ commutes with $\widehat{X}$, it commutes with every function of $\widehat{X}$.

Complete commuting set: A set $\mathscr{S}$ of operators is called *complete commuting set* if it is a commuting set and if any operator which commutes with every member of the set $\mathscr{S}$ can be written as function of the operators in the set $\mathscr{S}$.

The concept of complete commuting set is important in choosing a basis and working with representations as against with abstract vector space.

For more details, we refer the reader to
T. F. Jordan,*Linear Operators for Quantum Mechanics*, John Wiley and Sons, New York(1969)

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