Mathematical Physics: Some Tricks

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1 Calculating Determinants by Chio's Rule

Although many students may be using *Mathematica* for calculating determinants these days, there is merit in calculating lower order determinants 'by hand'. After all, one cannot not stop teaching numerical evaluation just because calculators are available.

I have found the following rule very useful in calculating numerical determinants (particularly of size 4×4 or higher). I learnt it from the book : *Applied Mathematics for Engineers and Physicists* by Louis A. Pipes, a very good book I still recommend to students.

The rule allows you to reduce the determinant size by one at each step and produce only one term (and not many terms as will happen in expanding by row or column). This makes the rule particularly convenient in evaluating numerical determinants.

Suppose the determinant to be evaluated is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & . & a_{1j} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2j} & . & a_{2n} \\ . & . & . & . & . & . \\ a_{i1} & a_{i2} & . & a_{ij} & . & a_{in} \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & a_{nj} & . & a_{nn} \end{vmatrix}$$

The Chio rule says:
Choose any non-zero element, say a_{ij}, called the *pivotal element*. Then, omit the i-th row and j-th column and write the $(n-1)\times(n-1)$ determinant by replacing each element by its product with the pivotal element and subtracting from it the product of elements at which the row and column of this element cut the i-th row and j-th column.

$$|A| = (-1)^{i+j}(a_{ij})^{n-2} \begin{vmatrix} a_{11}a_{ij} - a_{i1}a_{1j} & a_{12}a_{ij} - a_{i2}a_{1j} & . & . & a_{1n}a_{ij} - a_{in}a_{ij} \\ a_{21}a_{ij} - a_{i1}a_{2j} & a_{22}a_{ij} - a_{i2}a_{2j} & . & . & a_{2n}a_{ij} - a_{in}a_{2j} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1}a_{ij} - a_{i1}a_{nj} & a_{n2}a_{ij} - a_{i2}a_{nj} & . & . & a_{nn}a_{ij} - a_{in}a_{nj} \end{vmatrix}$$

The terms to be substituted for a given element v is easily seen in the diagram below:

$$\begin{vmatrix} . & | & . & | & . & . \\ - & v & - & a & - & - \\ . & | & . & | & . & . \\ - & b & - & a_{ij} & - & - \\ . & | & . & | & . & . \\ . & | & . & | & . & . \end{vmatrix} \qquad v \to va_{ij} - ab$$

Proof:
Assume that the i-th row has, apart from a_{ij}, other elements non-zero. This, as we shall immediately see is not essential. We can replace the zero elements in the i-th row with an infinitesimal number ϵ and later take that to be zero.

Pull out all elements from the i-th row as factors by dividing the corresponding columns :

$$|A| = a_{i1}a_{i2}\ldots a_{ij}\ldots a_{in} \begin{vmatrix} \frac{a_{11}}{a_{i1}} & \frac{a_{12}}{a_{i2}} & . & \frac{a_{1j}}{a_{ij}} & . & \frac{a_{1n}}{a_{in}} \\ \frac{a_{21}}{a_{i1}} & \frac{a_{22}}{a_{i2}} & . & \frac{a_{2j}}{a_{ij}} & . & \frac{a_{2n}}{a_{in}} \\ . & . & . & . & . & . \\ 1 & 1 & . & 1 & . & 1 \\ . & . & . & . & . & . \\ \frac{a_{n1}}{a_{i1}} & \frac{a_{n2}}{a_{i2}} & . & \frac{a_{nj}}{a_{ij}} & . & \frac{a_{nn}}{a_{in}} \end{vmatrix}$$

and subtract the j-th column from each other column to get

$$|A| = a_{i1}a_{i2}\ldots a_{ij}\ldots a_{in} \begin{vmatrix} \frac{a_{11}}{a_{i1}} - \frac{a_{1j}}{a_{ij}} & \frac{a_{12}}{a_{i2}} - \frac{a_{1j}}{a_{ij}} & . & \frac{a_{1j}}{a_{ij}} & . & \frac{a_{1n}}{a_{in}} - \frac{a_{1j}}{a_{ij}} \\ \frac{a_{21}}{a_{i1}} - \frac{a_{2j}}{a_{ij}} & \frac{a_{22}}{a_{i2}} - \frac{a_{2j}}{a_{ij}} & . & \frac{a_{2j}}{a_{ij}} & . & \frac{a_{2n}}{a_{in}} - \frac{a_{2j}}{a_{ij}} \\ . & . & . & . & . & . \\ 0 & 0 & . & 1 & . & 0 \\ . & . & . & . & . & . \\ \frac{a_{n1}}{a_{i1}} - \frac{a_{nn}}{a_{in}} & \frac{a_{n2}}{a_{i2}} - \frac{a_{nn}}{a_{in}} & . & \frac{a_{nj}}{a_{ij}} & . & \frac{a_{nn}}{a_{in}} - \frac{a_{nn}}{a_{in}} \end{vmatrix}.$$

Now expand the determinant and take all the factors inside (including an extra $(n-2)$ factors of a_{ij}) to multiply to $(n-1)$ columns, to get the result.

This rule is particularly useful when the chosen pivotal element happens to be equal to 1. This avoids keping trach of $(a_{ij})^{n-2}$ factors.

As an example we calculate the determinant

$$|A| = \begin{vmatrix} 7 & 5 & 2 & 3 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 3 & 5 \\ 8 & 1 & 5 & 2 \end{vmatrix}.$$

Take the 1 in the second row as the pivotal element then, Chio's rule gives

$$|A| = -\begin{vmatrix} 7-4 & 5-6 & 3-0 \\ 0-6 & 1-9 & 5-0 \\ 8-10 & 1-15 & 2-0 \end{vmatrix} = -\begin{vmatrix} 3 & -1 & 3 \\ -6 & -8 & 5 \\ -2 & -14 & 2 \end{vmatrix}.$$

Take the minus sign inside and multiply to the second column to have 1 in the first row as a pivotal element. This gives

$$|A| = -\begin{vmatrix} -6-24 & 5-24 \\ -2-42 & 2-42 \end{vmatrix} = -\begin{vmatrix} -30 & -19 \\ -44 & -40 \end{vmatrix} = -364.$$